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Theoretical Computer Science

The Locally Finite Fixpoint and its properties

A study of the semantics of finitely generated state and equation systems

Master Thesis in Computer Science

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Abstract

Coalgebras are a generic and elegant way to talk about state and equation-systems. The unique coalgebra homomorphism into the final coalgebra can be understood as the "behaviour", "semantics", or "solution" of the corresponding system.

In the present work we develop a uniform framework to talk about the subcoalgebra of the final coalgebra, that captures precisely the behaviours of finitely generated systems, that is coalgebras with a finitely generated carrier. The induced coalgebra – the *locally finite fixpoint* (LFF) – is a fixpoint of the functor and has multiple equivalent characterizations. (E.g. as the final *locally finitely generated* coalgebra or as the initial iterative algebra for equation systems).

At first, all applications of the already well-investigated rational fixpoint (the coalgebra final for coalgebras with a finitely presented carrier) that can be found in the literature are examples for the LFF. That is because in the considered categories finitely generated and finitely presented objects coincide. However, the new examples of the LFF we list, are not necessarily instances of the rational fixpoint as the finitely generated objects are strictly more.

As examples we present: 1. The context-free languages on Σ as the LFF of a certain lifting of $2 \times (-)^{\Sigma}$ to the category of Σ-pointed idempotent semi-rings; 2. (The monad of) Courcelle's algebraic trees as the LFF for an appropriate functor on the category of pointed finitary monads.

Contents

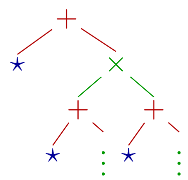
1 Introduction

Coalgebras offer a generic way to talk about various fundamental notions of computer science, including streams, possibly infinite data structures, recursive program schemes, automata, and other machines from the Chomsky hierarchy.

In most of these examples it plays a crucial role, that the considered instances can be described in a finite way. The stack of a stack machine may grow arbitrarily during the execution, so stack machines have infinitely many configurations. Even so, it is an important restriction to have only finitely many states. If we consider the transition table of a stack machine as its *program*, the restriction can be rephrased as the finiteness of the program.

The condition of such a "program" being finite gets more obvious in terms of equation systems: for a signature Σ of "givens", there are different notions of equation systems, for example flat equation systems and recursive program schemes. An example for a flat equation system over the signature $\Sigma = \{\star/0, \times/2, +/2\}$, i.e. a constant symbol $\star$ and binary symbols $\times$ and $+$, is

$$\begin{aligned} u &= \star \\ v &= u + w \\ w &= v \times v \end{aligned} \quad \rightsquigarrow$$

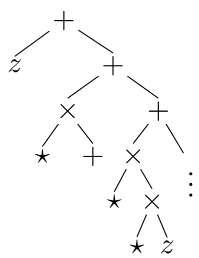


The tree on the right-hand side is the uninterpreted solution of v and has the uninterpreted solution of u and of w as its left and right subtree respectively. We can verify that these three trees are indeed a solution for the above equation system in the usual way: if we substitute v, u, and w by their respective proposed solution in the equations we obtain identities. Conversely, one obtains the solution by unravelling the equations.

If we require the number of equations to be finite, the solutions we obtain are precisely the rational Σ-trees: possibly infinite Σ-trees with only finitely many subtrees (up to isomorphism).

An example of a recursive program scheme, for the same Σ, is

$$\varphi(z) = z + \varphi(z \times \star) \quad \rightsquigarrow$$



where the uninterpreted solution of $\varphi(z) = z + \varphi(z \times \star)$ is the right-hand tree and obtained by successively unfolding the definition of φ in the program scheme. If we require our program signature to be of finite arity, the solutions we obtain are the so called algebraic Σ-trees, a proper superclass of the rational Σ-trees.

In both kinds of equation systems, the restriction of finiteness plays a crucial role in order to get a meaningful notion of solutions instead of just all Σ-trees.

With this thesis, we provide a uniform framework to talk about finite state and equation systems that can be described coalgebraically.

Outline of this Thesis In Section 2 we motivate this thesis by summarizing the existing work on coalgebras with finite carrier and by briefly discussing problems of the existing approach. Before solving these problems, we recall some basic definitions needed for this work in Section 3. Having filled our toolbox, we are ready to develop the locally finite fixpoint (LFF) in Section 4 and see that it indeed meets the requirements from Section 2. In Section 5, we see that the LFF can be applied to scenarios in which previous approaches failed, including the examples of stack machines and recursive program schemes. We conclude by suggesting further applications of the LFF in Section 6.

2 Motivation and Existing Work

With this thesis, the goal is to give a uniform framework for the semantics of finitely generated systems. State and equation systems can be modelled elegantly as coalgebras. Let us first recap some basic results and examples of coalgebras. Furthermore, let us shortly look at an existing approach of modelling finiteness for coalgebras, let us see its limitations and downsides, which we want to solve with the present work. After that, we will dive into the precise definitions in the following chapters in order to see that the locally finite fixpoint has indeed the desired properties.

2.1 Coalgebras

For a category $\mathcal{B}$ and an endofunctor $H : \mathcal{B} \to \mathcal{B}$, the theory of H-coalgebras gives us a categorical framework to define state and equation systems. The functor H hereby describes the structure of the successors for state systems and the signature of the equations for equation systems. Given an object C of $\mathcal{B}$, the actual state transitions, or the actual equations, are defined by a morphism

$$C \xrightarrow{c} HC$$

and we say that (C, c) is a coalgebra.

The *final* H-coalgebra $(\nu H, \tau : \nu H \to H\nu H)$ induces for each coalgebra (C, c) a unique behaviour, i.e. a unique coalgebra homomorphism into the final coalgebra. In other words, the final coalgebra contains all behaviours which can be described by arbitrary H-coalgebras (C, c). Popular examples include the following:

- For $\mathcal{B} = \mathsf{Set}$, and $HX = \mathbb{N} \times X$, H-coalgebras describe transition systems with natural numbers on the edges. The final H-coalgebra is carried by the set of streams of natural numbers $\mathbb{N}^\omega$, with the pair of head and tail as the coalgebra structure $\langle \mathrm{hd}, \mathrm{tl} \rangle : \mathbb{N}^\omega \to \mathbb{N} \times \mathbb{N}^\omega$.
- In Set again, for some signature functor H_Σ corresponding to a signature of function symbols Σ, an H_Σ-coalgebra carried by V can be regarded as a system of equations of the form
$$v = T, \quad \text{with } v \in V,$$
where T is a Σ-term of height 1 involving variables from V. The final H-coalgebra contains all possibly infinite trees for Σ and the unique coalgebra homomorphism for an H_Σ-coalgebra maps each variable to its solution.
- For $\mathcal{B} = \mathsf{Set}$, an input alphabet Σ, and $HX = 2 \times X^\Sigma$, the H-coalgebras $C \to 2 \times C^\Sigma$ can be considered as automata. The final H-coalgebra contains all formal languages over Σ. Here, the left-hand component of the structure τ describes whether a formal language contains the empty word and the right-hand component gives for each $\sigma \in \Sigma$ the according left-derivation of languages.
- For a non-Set example, consider the category of nominal sets $\mathcal{B} = \mathsf{Nom}$ and $HX = \mathcal{V} + [\mathcal{V}]X + X \times X$. Here, $\mathcal{V}$ is a countable set of variable names and $[\mathcal{V}]$ the binding functor, see [GP99] for more details. By [KPSdV13], the final H-coalgebra contains the possibly infinite λ-terms with finitely many free variables modulo α-equivalence.

2.2 Notions of Finiteness

Note that until now, there is no restriction on the carrier C of a coalgebra. But in practice, computer scientists normally talk about *finite* structures or, more generally, about structures that can be described in a finite way.

In category theory, there are two notions of finiteness of objects: the property of being *finitely presentable* (f.p.) and of being *finitely generated* (f.g.). While every finitely generated object is finitely presentable the converse direction does not hold in general. But the finitely generated objects are always the closure of the finitely presentable ones under strong quotients.

- In Set, finitely presentable and finitely generated coincide and precisely denote the finite sets.
- For a field K, in the category of K-vector spaces, the two notions coincide like in Set and characterize finite dimensional K-vector spaces.
- In the category of Monoids, finitely presented means being generated by finitely many generators and finitely many relations, whereas finitely generated monoids have finitely many generators but possibly infinitely many relations. And f.g. monoids that are not f.p. indeed exist, see [CRRT96, Example 4.5] or any other example of a f.g. group that is not finitely presentable.
- More generally, for a finitary monad $T : \mathsf{Set} \to \mathsf{Set}$ (which corresponds to an algebraic theory), in the Eilenberg-Moore-category Set^T the finitely presentable objects are those algebras that can be presented by finitely many generators and relations, whereas the finitely generated objects are those algebras presented by finitely many generators.

2.3 The Rational Fixpoint

In the past, the *rational fixpoint* has been developed to talk about behaviours that can be described by coalgebras with a finitely presentable carrier. More precise, the rational fixpoint of a functor $H : \mathcal{B} \to \mathcal{B}$ is the coalgebra $(\varrho H, r)$ which is uniquely determined by the following universal property [AMV06]:

$$\begin{tikzcd}
\text{For every } X \arrow[r, "x"] \arrow[d, dashed, "\exists! x^\dagger"'] & HX \text{ with } X \text{ f.p.} \arrow[d, "Hx^\dagger"] \\
\varrho H \arrow[r, "r"] & H\varrho H
\end{tikzcd}$$

Similar to the final coalgebra, ϱH is a fixpoint, i.e. the structure r is an isomorphism. Though ϱH is not finitely presentable, $(\varrho H, r)$ is *locally finitely presentable* (lfp). A coalgebra (Y, y) is called locally finitely presentable, if each morphism $f : S \to Y$ with S finitely presentable factors into some $f' : S \to P$ and some coalgebra morphism $h : (P, p) \to (Y, y)$ and f' must be essentially unique. In [Mil10], the notion of lfp coalgebra was introduced and it was proven that lfp coalgebras are precisely the filtered colimits of coalgebras with f.p. carrier. With that definition, $(\varrho H, r)$ is the *final* lfp coalgebra.

The rational fixpoint applies to many previous examples, involving the following ones:

- For signature functors H_Σ on Set, $\varrho\Sigma$ contains all regular Σ-trees of Elgot [Elg75] (see also Courcelle [Cou83]). The regular Σ-trees are those having only finitely many different subtrees up to isomorphism (see Ginali [Gin79]).

- This applies also to the Set endofunctor $HX = \mathbb{N} \times X$: ϱH denotes all eventually periodic streams of natural numbers.
- For $HX = 2 \times X^{\Sigma}$, for some input alphabet Σ, ϱH contains all regular languages over Σ, see [AMV06]. If one considers coalgebras of the form

  $$X \to HTX,$$

  where T is some monad, one can even cover variations of ordinary automata, e.g. partial, non-deterministic, or probabilistic automata. Then the fixpoints ϱH and $\varrho(HT)$ can be used to derive expression calculi for language equivalence for the respective automaton flavour, see [BMS13].

In all these examples, the rational fixpoint extracts a subcoalgebra out of the final coalgebra. This is the case, because if finitely presentable objects are closed under strong quotients, the rational fixpoint is just the image in the final coalgebra of all coalgebras with f.p. carrier.

The benefit of ϱH being a subcoalgebra is: if two states of two coalgebras are identified in the final coalgebra – i.e. if they have *identical behaviour* – they are identified in ϱH as well. However, this is not the case in general; in particular in some relevant examples.

- For $HX = B \times X^{\Sigma}$ on Set and the non-deterministic stack monad T (see [Gon13]; we recall it in Subsection 5.3.3), coalgebras of the form

  $$X \to HTX$$

  precisely describe non-deterministic push-down automata [GMS14], where $B \subseteq 2^{\Gamma^*}$ is a set of functions which map stack configurations (for some stack alphabet Γ) to output values from 2 considering only the topmost elements from the stack.

  When constructing their image in the final coalgebra νH, one has to lift those coalgebras to the Eilenberg-Moore category Set^T in order to "convert" them into ordinary H-coalgebras. But in Set^T, it is unknown whether the classes f.g. and f.p. objects coincide. So the rational fixpoint of the lifting of H may not be a subcoalgebra of νH.
- Recursive program schemes can be modelled as coalgebras in a certain category of monads. Their solutions, i.e. their image in the terminal coalgebra, are precisely the so called *algebraic trees*. We have a similar situation here as the rational fixpoint of the corresponding functor might not be a subobject of the terminal coalgebra but its image describes the algebraic trees.

In those examples, there are behaviours which are not recognized being the same by ϱH, see [BMS13, Example 3.15].

In the above examples, it is not clear whether (and we believe, that it is not true that) the rational fixpoint identifies all behavioural equivalent states of f.p. carried coalgebras. See [BMS13, Example 3.15] for an example where this failure of the rational fixpoint to be a subcoalgebra of the final coalgebra was demonstrated.

In all such examples, a solution is to factorize the unique $f : \varrho H \to \nu H$ with the (strong epi,mono)-factorization system of the base category, which lifts to coalgebras. This gives us a subcoalgebra of νH that contains the behaviours of all f.p. carried coalgebras as desired. This subcoalgebra of νH already appears explicitly in Rob Myers' PhD thesis [Mye11] (under the name "rational image") and in the coalgebraic treatment of algebraic trees in [AMV11a]. However, this coalgebra and its properties have not been investigated in their own right. This is the goal of this thesis: we will define this coalgebra, show that it is a fixpoint and two characterizations by universal properties. We explain this in the next subsection a bit more in detail.

2.4 The Locally Finite Fixpoint

While the rational fixpoint ϱH is based on f.p. carried coalgebras, we will start with coalgebras with a *finitely generated* carrier instead. So we aim for a uniform framework to talk about f.g. coalgebras, and their behaviour, which can be used in all applications of the rational fixpoint as well as in scenarios in which the rational fixpoint is not a subcoalgebra of the final coalgebra.

Furthermore, we adapt the notion of an lfp coalgebra to that of a *locally finitely generated* (lfg) coalgebra. By that we get results analogous to properties of the rational fixpoint:

- The final lfg coalgebra is a fixpoint for H, and hence deserves the title *locally finite fixpoint.*
- The lfg coalgebras are precisely filtered colimits (even directed unions) of coalgebras with f.g. carrier.
- The *final* lfg coalgebra $(\vartheta H, \ell : \vartheta H \to H\vartheta H)$ is uniquely determined by the property:

$$\begin{tikzcd} \text{For every } X \arrow[r, "x"] \arrow[d, dashed, "\exists! x^\dagger"'] & HX \text{ with } X \text{ f.g.} \arrow[d, "Hx^\dagger"] \\ \vartheta H \arrow[r, "\ell"] & H\vartheta H \end{tikzcd}$$

  Note that since all f.p. coalgebras are f.g., ϑH is also final for them.

- The inverse $\ell^{-1} : H\vartheta H \to \vartheta H$ has an algebraic meaning. An *equation morphism* e in an object A is a morphism

$$X \to HX + A,$$

  where X is finitely generated. If $(A, \alpha : HA \to A)$ is an algebra, a morphism $e^\dagger : X \to A$ is a solution of e if

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd}$$

  commutes. We say that an H-algebra A is fg-iterative if every equation morphism has a unique solution in A. The fg-iterative algebras form a category, and $(\vartheta H, \ell^{-1})$ is its initial object.

- Every lfp coalgebra is lfg.

But in addition to that, we get the desired properties, the rational fixpoint lacks:

- The lfg coalgebras are closed under homomorphisms carried by strong quotients.
- The locally finite fixpoint is a subcoalgebra of the final coalgebra.

This fixpoint is indeed the missing piece we were looking for: under some more assumptions, met by all examples so far, the locally finite fixpoint is the image of the rational fixpoint in the final coalgebra.

3 Preliminaries

We assume that the reader is familiar with basic categorical notions, as described in the first chapters of [Awo10], in particular the notions of category, functor, natural transformation, limit and colimit.

We write Set for the category of sets and functions; hom-sets of a category $\mathcal{C}$ are denoted by $\mathcal{C}(A, B)$ and the identity morphisms by id_X or simply by the object X. The identity functor on $\mathcal{C}$ is denoted by $\mathrm{Id}_{\mathcal{C}}$. The initial object of category is denoted by 0, the terminal object by 1.

For a diagram $D : \mathcal{D} \to \mathcal{C}$ the colimit is denoted by $\operatorname{colim} D$. If not defined otherwise, the colimit injections are called $\mathsf{in}_X : DX \to \operatorname{colim} D$; and for the binary coproduct by $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$. Furthermore, for a functor F we define the canonical morphism $\mathsf{can} : FX + FY \to F(X + Y)$ by $\mathsf{can} := F[\mathsf{inl}, \mathsf{inr}]$.

For $f : A \to C$ and $g : B \to C$, we denote by $[f, g] : A + B \to C$ the unique morphism induced by the universal property of the coproduct $A + B$. Dually, for $f : P \to A$ and $g : P \to B$, we write $\langle f, g\rangle : P \to A \times B$ for the unique morphism induced by the product $A \times B$.

Other basic notions are defined in this section, whereas more advanced definitions will be given in the following sections when they are needed. For a list of used symbols and notations – commonly known or specific to this thesis – see page 77.

Definition 3.1 (Coslice category)**.** For an object $C \in \mathcal{C}$, the coslice category $C/\mathcal{C}$ has as objects $\mathcal{C}$-morphisms with domain C and as morphisms from $(a : C \to A)$ to $(b : C \to B)$ all $\mathcal{C}$-morphisms $m : A \to B$ for which $m \cdot a = b$.

For example, the coslice category $1/\mathsf{Set}$ is the category of pointed sets.

3.1 Special Morphisms

Let us quickly recall the most basic definitions which will be needed later in this section.

Definition 3.2 (Monomorphism)**.** A morphism $m : A \to B$ is called a *monomorphism* (shortened to *mono*) or is said to be *monic* if for every pair of morphisms $g, h : C \to A$ with $m \cdot g = m \cdot h$ the equality $g = h$ holds. We write $m : A \rightarrowtail B$.

Definition 3.3 (Epimorphism)**.** A morphism $e : A \to B$ is called an *epimorphism* (shortened to *epi*) if for every pair of morphisms $g, h : B \to C$ with $g \cdot e = h \cdot e$ the equality $g = h$ holds. We write $e : A \twoheadrightarrow B$.

Similarly, we call a family of morphisms with a common codomain $(e_i : A_i \to B)_{i \in I}$ *jointly epic*, if for $g, h : B \to C$ the condition $\forall i \in I : g \cdot e_i = h \cdot e_i$ implies $g = h$.

Example 3.4.

- Colimit injections are jointly epic.
- If for a jointly epic family $c_i : A_i \to C$ there is a factorization
$$c_i = e \cdot b_i \quad \forall i \in I, \quad b_i : A_i \to B, \quad e : B \to C,$$
then e is epic.

Too see that, consider $g, h : C \to D$ with $g \cdot e = h \cdot e$. Multiplying each side with b_i for each $i \in I$ gives $g \cdot e \cdot b_i = h \cdot e \cdot b_i = g \cdot a_i = h \cdot a_i$, hence $g = h$, as the a_i are jointly epic.

Proposition 3.5. *If $\mathcal{C}$ has coproducts, then monomorphisms in the coslice category $C/\mathcal{C}$ are precisely the morphisms carried by a monomorphism.*

Proof. The direction "mono in $\mathcal{C}$ $\Rightarrow$ mono in $C/\mathcal{C}$" is easy. For the converse, consider a mono $m : (c_B : C \to B) \hookrightarrow (c_D : C \to D)$ in $C/\mathcal{C}$ and $f, g : A \to D$ with $mf = mg$:

$$\begin{tikzcd}
& C \arrow[dl, bend right, "\mathsf{inr}"'] \arrow[d, "c_B"] \arrow[dr, "c_D"] & \\
A + C \arrow[r, shift left, "{[f, c_B]}"] \arrow[r, shift right, "{[g, c_B]}"'] & B \arrow[r, "m"] & D \\
A \arrow[u, "\mathsf{inl}"] \arrow[ur, bend right, "f"] \arrow[ur, bend right=50, "g"'] & &
\end{tikzcd}$$

By the universal property of the coproduct $A + C$ we get two morphisms $A + C \to B$ which preserve the pointing, i.e. which both are morphisms $(\mathsf{inr} : C \to A + C) \to (c_B : C \to B)$ in $C/\mathcal{C}$ with $m \cdot [f, c_B] = m \cdot [g, c_B]$. As m is monic in $C/\mathcal{C}$, we get $[f, c_B] = [g, c_B]$ and hence $f = [f, c_B] \cdot \mathsf{inl} = [g, c_B] \cdot \mathsf{inl} = g$. $\square$

Proposition 3.6. *In the coslice category $C/\mathcal{C}$, the epimorphisms are precisely the morphisms carried by an epimorphism.*

Proof. The direction "epi in $\mathcal{C}$ $\Rightarrow$ epi in $C/\mathcal{C}$" is easy. For the converse, consider an epi $e : (c_A : C \to A) \twoheadrightarrow (c_B : C \to B)$ in $C/\mathcal{C}$ and $f, g : B \to D$ with $fe = ge$:

$$\begin{tikzcd}
& C \arrow[dl, "c_A"'] \arrow[d, "c_B"] \arrow[dr, "f \cdot e \cdot c_A = g \cdot e \cdot c_A =: c_D"] & \\
A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D
\end{tikzcd}$$

Both f and g preserve the pointing:

$$f \cdot e \cdot c_A = f \cdot c_B = g \cdot e \cdot c_A,$$

so $f = g$ as e is epic in $C/\mathcal{C}$. $\square$

Definition 3.7 (Orthogonal)**.** Two morphisms $e : A \to B$ and $m : C \to D$ are said to be *orthogonal*, written $e \perp m$, if for each pair of morphisms $g : A \to C$ and $h : B \to D$ with $m \cdot g = h \cdot e$ there exists a unique diagonal $d : B \to C$ such that the following square commutes:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dashed, "d"'] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

If $e \perp m$ holds, we say that e is left-orthogonal to m.

Definition 3.8 (Strong epimorphism)**.** An epimorphism e is said to be *strong*, if it is left-orthogonal to any monomorphism. I.e. for any mono m, it is $e \perp m$.

Proposition 3.9. *For a category $\mathcal{C}$ with coproducts, the strong epis in $C/\mathcal{C}$ are precisely the morphisms carried by a strong epi.*

Proof. It is easy to show that any strong epi-carried morphism e in $C/\mathcal{C}$ is indeed a strong epi: by Proposition 3.5 all monos m in $C/\mathcal{C}$ are mono-carried and so right-orthogonal to e in $\mathcal{C}$, which induces a diagonal that necessarily preserves the pointing, so $e \perp m$ in $C/\mathcal{C}$.

For the converse direction, consider a strong epi $e : (c_A : C \to A) \twoheadrightarrow (C_B : C \to B)$ in $C/\mathcal{C}$ together with a mono $m : D \rightarrowtail E$ in $\mathcal{C}$ and $g : A \to D$, $h : B \to E$ with $m \cdot g = h \cdot e$.

$$\begin{tikzcd}
& C \arrow[dl, "c_A"'] \arrow[dr, "c_B"] \arrow[ddl, bend right=60, "g \cdot c_A"'] \arrow[ddr, bend left=60, "h \cdot c_B"] & \\
A \arrow[rr, two heads, "e"] \arrow[d, "g"'] & & B \arrow[d, "h"] \\
D \arrow[rr, tail, "m"] & & E
\end{tikzcd}$$

Trivially, we can equip D and E with a pointing, directly making g and h morphisms of $C/\mathcal{C}$. As all parts of the above diagram commute, m is in $C/\mathcal{C}$ as well and monic there. So we get a unique diagonal $d : B \to D$ that preserves the pointing: $d \cdot c_B = g \cdot c_A$. The latter is not a restriction, i.e. for any other diagonal $\bar{d}$ in $\mathcal{C}$, we have $\bar{d} \cdot c_B = \bar{d} \cdot e \cdot c_A = g \cdot c_A$, hence $\bar{d} = d$. $\square$

Definition 3.10 ((E, M)-factorization system)**.** Let E and M be classes of morphisms in a category $\mathcal{C}$, then (E, M) is called a *factorization system* in $\mathcal{C}$ if:

1. each of E and M is closed under composition with isomorphisms.
2. each morphism f has a factorization $f = m \cdot e$ with $e \in E$ and $m \in M$.
3. for all $e \in E$ and $m \in M$, $e \perp m$.

Note that these conditions imply that the (E, M)-factorization of a morphism is unique up to isomorphism. Consider two factorizations $m' \cdot e' = f = m \cdot e$ of $f : A \to C$ through intermediate objects B and B':

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e'"'] & B \arrow[d, "m"] \arrow[dl, dashed, bend right=15, "i"'] \\
B' \arrow[r, "m'"] \arrow[ur, dashed, bend right=15, "j"'] & C
\end{tikzcd}$$

As $e \perp m'$, a unique $i : B \to B'$ is induced as well as a unique $j : B' \to B$ by $e' \perp m$. From here, it is easy to follow that i, j are isomorphisms and inverse to each other, i.e. that the factorization is unique.

As an example, consider the (epi,mono)-factorizations of a $f : A \to C$ in Set: this is precisely the factorization of f through its image $\mathsf{Im}(f) \subseteq C$. This is generalized to (strong epi,mono)-factorizations in other categories.

Definition 3.11 (Image)**.** In a category with (strong epi,mono)-factorizations and a morphism $f : A \to B$, the image of the morphism f is the intermediate object $\mathsf{Im}(f)$ of its (strong epi,mono)-factorization together with the epi $A \twoheadrightarrow \mathsf{Im}(f)$. I.e. for

$$f \equiv \left(\; A \xrightarrow{f_e} \!\!\!\!\twoheadrightarrow I \rightarrowtail^{f_m} B \; \right)$$

The image of f is $\mathsf{Im}(f) := I$, together with $f_e : A \twoheadrightarrow I$.

Recall that for morphisms f, g, if $g \cdot f$ is an epimorphism, then g is an epi. A similar property holds for strong epis:

Proposition 3.12. *If for two morphisms $f : A \to B$ and $g : B \to C$ the composition $g \cdot f$ is a strong epi, then g is a strong epi.*

Proof. At first, g must be an epimorphism. Consider a mono $m : E \to F$ and arbitrary $k : B \to E$, $\ell : C \to F$ for which $m \cdot k = \ell \cdot g$. So all parts of the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, two heads, "g \cdot f"] \arrow[dr, "k \cdot f"'] & B \arrow[r, two heads, "g"] \arrow[d, "k"'] & C \arrow[d, "\ell"] \\
& D \arrow[r, tail, "m"'] & E
\end{tikzcd}$$

commute. As $g \cdot f$ is a strong epi, a unique diagonal $d : C \to D$ is induced for which $m \cdot d = \ell$ and $d \cdot g \cdot f = k \cdot f$. Furthermore, we have

$$m \cdot d \cdot g = \ell \cdot g = m \cdot k.$$

This implies $d \cdot g = k$, as m is monic. For any other other $\tilde{d} : C \to D$ with $m \cdot \tilde{d} = \ell$ and $\tilde{d} \cdot g = k$, we have $\tilde{d} \cdot g \cdot f = k \cdot f$ too and so a diagonal for the outer square. Hence $\tilde{d} = d$, i.e. d is the unique diagonal and g a strong epi. ☐

Proposition 3.13. *For some diagram $D : I \to \mathcal{C}$ with objects $Di = C_i$, $i \in I$, let $(c_i : C_i \to C)$ be a colimit cocone. Then the morphism $[c_i] : \coprod C_i \to C$ from the coproduct is a strong epi.*

Proof. Assume arbitrary $f : \coprod C_i \to A, g : C \to D, m : A \rightarrowtail B$ with $m \cdot f = g \cdot [c_i]$. As colimit injections are jointly epic, we already know that $[c_i]$ is an epimorphism. Furthermore,

$$\begin{tikzcd}
C_j \arrow[r, "\mathsf{in}_j"] \arrow[rr, bend left, "c_j"] & \coprod C_i \arrow[r, two heads, "{[c_i]}"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
& A \arrow[r, tail, "m"] & B
\end{tikzcd}$$

commutes for any $j \in I$. Here, the in_j denote the coproduct injections. For any connecting morphism $a : C_j \to C_k$ of the diagram,

$$m \cdot f \cdot \mathsf{in}_k \cdot a = g \cdot [c_i] \cdot \mathsf{in}_k \cdot a = g \cdot c_k \cdot a = g \cdot c_j = g \cdot [c_i] \cdot \mathsf{in}_j = m \cdot f \cdot \mathsf{in}_j.$$

As m is a mono, we have $f \cdot \mathsf{in}_k \cdot a = f \cdot \mathsf{in}_j$. In other words, $(f \cdot \mathsf{in}_i : C_i \to A)_{i \in I}$ forms a cocone. Therefore, a unique morphism $d : C \to A$ is induced with $d \cdot [c_i] \cdot \mathsf{in}_j = f \cdot \mathsf{in}_j$ for all $j \in I$ and by the in_j being jointly epic, equivalently with $d \cdot [c_i] = f$. The other triangle also commutes:

$$m \cdot d \cdot [c_i] = m \cdot f = g \cdot [c_i] \quad \overset{[c_i] \text{ epi}}{\Longrightarrow} \quad m \cdot d = g$$

Any other $\tilde{d} : C \to D$ making the two triangles commute implies $\tilde{d} \cdot [c_i] \cdot \mathsf{in}_j = f \cdot \mathsf{in}_j$ for any $j \in I$. We have $\tilde{d} = d$ directly, as d was the unique cocone morphism $C \to A$. ☐

3.2 Special Objects

Using monos and epis, we can describe special kinds of relationships between objects as follows.

Definition 3.14 (Quotient, strong quotient)**.** A (strong) quotient of some object X is an isomorphism class of (strong) epimorphisms

$$q : X \to Q.$$

We say that two (strong) epis $q : X \to Q$, $q' : X \to Q'$ are isomorphic if there is some iso $i : Q \to Q'$ such that $i \cdot q = q'$. Normally, we consider just one representative $q : X \to Q$ to denote the quotient or say Q is the quotient of X, if q is clear for the context.

Definition 3.15 (Subobject)**.** A *subobject* of an object X is an isomorphism class of monomorphism

$$m : A \to X.$$

We say that two monos $m : A \to X$, $m' : A' \to X$ are isomorphic is there is some iso $i : A \to A'$ such that $m = m' \cdot i$. Normally, we consider just one representative $m : A \to X$ to denote the subobject.

3.3 Adjunctions

Recall the definition of adjoint functors and its many equivalent formulations from [Awo10]. We basically use that an adjunction $F \dashv U$ consists of two functors $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ such that there is a natural isomorphism $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, UY)$, i.e. a one-to-one correspondence

$$\frac{FX \longrightarrow Y}{X \longrightarrow UY} \tag{3.1}$$

which is natural in both X and Y. In most cases, we denote the unit of this adjunction by $\eta : \mathrm{Id} \to UF$, where Id denotes the identity functor. Also recall this fact:

Proposition 3.16. *Epimorphisms are preserved by left adjoint functors. Dually, right adjoints preserve monomorphisms.*

In this work, we need to work a lot with strong epimorphisms, which are preserved as well:

Proposition 3.17. *For an adjunction $F \dashv U$, the left adjoint $F : \mathcal{C} \to \mathcal{D}$ preserves strong epis.*

Proof. As we already know from Proposition 3.16, Fe is epic. So assume the commuting square

$$\begin{tikzcd} FA \arrow[r, two heads, "Fe"] \arrow[d, "g"'] & FB \arrow[d, "h"] \\ C \arrow[r, tail, "m"] & D \end{tikzcd} \tag{3.2}$$

in $\mathcal{D}$ where m is monic. Applying $U : \mathcal{D} \to \mathcal{C}$ to this square gives a commuting square in $\mathcal{C}$.

Extend it by the unit components $\eta_A : A \to UFA$ and $\eta_B : B \to UFB$ as follows:

$$
\begin{tikzcd}
A \arrow[rrr, two heads, "e"] \arrow[dr, "\eta_A"] \arrow[ddr, bend right, "g'"'] & & & B \arrow[dl, "\eta_B"'] \arrow[ddl, bend left, "h'"] \\
\arrow[r, phantom, "\text{(ii)}"] & UFA \arrow[r, two heads, "UFe"] \arrow[d, "Ug"'] \arrow[u, phantom, "\text{(i)}"] & UFB \arrow[d, "Uh"] \arrow[r, phantom, "\text{(iii)}"] & \\
& UC \arrow[r, tail, "Um"'] & UD &
\end{tikzcd}
\tag{3.3}
$$

Let us check where the diagram parts come from and why they commute.

(i) This commutes, because $\eta : \mathrm{Id}_{\mathcal{C}} \to UF$ is a natural transformation.

(ii) By the adjunction (3.1), $FA \xrightarrow{g} C$ corresponds to some $g' : A \to UC$, which factors through UFA.

(iii) By (3.1), $FB \xrightarrow{h} D$ corresponds to some $h' : B \to UD$, which factors through UFB.

So the outer square commutes and a unique diagonal $d' : B \to UC$ is induced. The adjunction induces some $d : FB \to C$ in $\mathcal{D}$ with $Ud \cdot \eta_B = d'$. This d can be proven to be the desired diagonal of the square in $\mathcal{D}$. Therefore, assume some arbitrary morphism $u : FB \to C$ in $\mathcal{D}$:

$$
\begin{array}{cccc}
 & \multicolumn{3}{c}{u \text{ is a diagonal for (3.2)}} \\
\Leftrightarrow & u \cdot Fe = g & \wedge & m \cdot u = h \\
\overset{(3.1)}{\Leftrightarrow} & Uu \cdot UFe \cdot \eta_A = g' & \wedge & Um \cdot Uu \cdot \eta_B = h' \\
\overset{\text{(i)}}{\Leftrightarrow} & Uu \cdot \eta_B \cdot e = g' & \wedge & Um \cdot Uu \cdot \eta_B = h' \\
\Leftrightarrow & \multicolumn{3}{c}{Uu \cdot \eta_B \text{ is a diagonal for the outer square in (3.3)}}
\end{array}
$$

Reading the equivalences from bottom to top for $u := d$ we see that d is a diagonal for (3.2). For any other diagonal $\tilde{d} : FB \to C$, reading the equivalences from top to bottom with $u := \tilde{d}$ gives $U\tilde{d} \cdot \eta_B = Ud \cdot \eta_B$. So by the adjunction we have $\tilde{d} = d$, i.e. d is indeed the unique diagonal. $\square$

4 The Locally Finite Fixpoint

Now, we are ready to develop the theoretical core of this thesis. This includes properties and the construction of the *locally finite fixpoint*. To have a meaningful notion of finiteness, we restrict to *locally finitely presentable (lfp)* categories.

4.1 Basics on Locally Finitely Presentable Categories

The rough idea of an locally finitely presentable (lfp) category is to have a subclass of *finitely presentable* objects which represent the basic parts every other object is assembled from. Before making this precise in Definition 4.5, which require further auxiliary definitions, let us look at some examples for lfp categories:

Example 4.1.

- In the lfp category Set, the finitely presentable objects are the finite sets. Furthermore, every set is the union of its finite subsets.
- Similarly, the category of posets and of graphs respectively is lfp and finitely presentable coincides with finiteness.
- For a field K, the category of K-vector spaces and linear maps Vec_K is lfp and the finite dimensional vector spaces are the finitely presentable objects.
- For any lfp category $\mathcal{C}$, the category of finitary endofunctors (see Definition 4.4 later) is lfp. For the case of Set-endofunctors, the finitely presentable ones are precisely quotients of polynomial functors H_Σ, where Σ is a finite signature and H_Σ the corresponding polynomial functor.
- For any lfp category $\mathcal{C}$ and an arbitrary object C, the coslice category $C/\mathcal{C}$ is lfp and the finitely presentable objects are objects $m : C \to B$ with B finitely presentable in $\mathcal{C}$, see [AR94, Corollary 2.44 and proof of Theorem 2.43].

Now follows, the precise definition, starting with the generalization of "assembling objects together". For a more comprehensive presentation of lfp categories, see [AR94].

Definition 4.2 (filtered category, filtered diagram)**.** A category $\mathcal{D}$ is called *filtered* if

- $\mathcal{D}$ is non-empty.
- for each two objects A, B there exists an object C and morphisms $A \to C$ and $B \to C$.
- for each pair $g, g' : A \to B$ there exists a morphism $f : B \to C$ with $f \cdot g = f \cdot g'$.

A diagram $D : \mathcal{D} \to \mathcal{B}$ is called filtered if $\mathcal{D}$ is filtered.

Definition 4.3 (directed category, directed diagram)**.** A category $\mathcal{D}$, as well as a diagram $D : \mathcal{D} \to \mathcal{B}$ is called *directed* if $\mathcal{D}$ is a partially ordered set $(I, \leq)$ in which each pair of elements has an upper bound. For a partially ordered set $(I, \leq)$ and two objects $i, j \in I$, we denote the unique morphism from i to j, if it exists, simply as $i \to j$.

Definition 4.4. A *filtered (directed) colimit* is the colimit of a filtered (directed) diagram $D : \mathcal{D} \to \mathcal{B}$. A functor $F : \mathcal{B} \to \mathcal{C}$ is called *finitary* if it preserves filtered colimits $D : \mathcal{D} \to \mathcal{B}$, i.e.

$$F \operatorname{colim} D \cong \operatorname{colim} FD.$$

Preserving filtered colimits is equivalent to preserving directed colimits [AR94, Theorem 1.5f]. An object $B \in \mathcal{B}$ is called finitely presentable or f.p. for short, if its hom-functor $\mathcal{B}(B, -)$ preserves filtered colimits.

Definition 4.5. A category $\mathcal{B}$ is called *locally finitely presentable* or *lfp* for short, if it is cocomplete and has a set $\mathcal{B}_{fp}$ of finitely presentable objects that every object from $\mathcal{B}$ is a directed colimit of objects from $\mathcal{B}_{fp}$.

Another notion of finiteness more relevant to this thesis is as follows.

Definition 4.6 (finitely generated, [AR94, Definition 1.1 and 1.67])**.** An object X of a category is called *finitely generated*, shortened f.g., provided that $\mathcal{C}(X, -)$ preserves directed colimits of monomorphisms, i.e. directed colimits where the connecting morphisms in $\mathcal{C}$ are monomorphisms.

More concretely, $X \in \mathcal{C}$ is *finitely generated* if for each diagram $D : (I, \leq) \to \mathcal{C}$ of monos and each colimit cocone $(c_i : Di \to C)_{i \in I}$ the morphisms

$$\big(\mathcal{C}(X, c_i) : \mathcal{C}(X, Di) \to \mathcal{C}(X, C)\big)_{i \in I}$$

form a colimit of the diagram $\mathcal{C}(X, D-) : (I, \leq) \to \mathsf{Set}$.

Theorem 4.7. *X is finitely generated iff for each directed diagram of monos*

$$D : (I, \leq) \to \mathcal{C},$$

for each colimit cocone $D_i \xrightarrow{c_i} C (= \operatorname{colim} D)$, $i \in I$, and for each morphism $f : X \to C$, there exists i such that

1. *f factors through c_i, i.e. $f = c_i \cdot g$ $(i \in I)$ for some $g : X \to D_i$,*
2. *the factorization is essentially unique in the sense that if $f = c_i \cdot g'$, then $g = g'$.*

Proof. For the direction "⇒", we have the colimit cocone

$$\big(\mathcal{C}(X, c_i) : \mathcal{C}(X, Di) \to \mathcal{C}(X, C)\big)_{i \in I}.$$

As $f \in \mathcal{C}(X, C)$ and as the colimit injections c_i are jointly epic, i.e. surjective in Set, there is some $i \in I$ with $g : X \to D_i$ and $\mathcal{C}(X, c_i)\ (g) = f$; hence $c_i \cdot g = f$.

For essential uniqueness, assume another $g' : X \to Di$ with $c_i \cdot g' = f$. But $\mathcal{C}(X, c_i)\ (g) = \mathcal{C}(X, c_i)\ (g')$ for a colimit in Set implies that there is some $i \to j$ in I with

$$\mathcal{C}(X, D(i \to j))\ (g) = \mathcal{C}(X, D(i \to j))\ (g') \;\Leftrightarrow\; D(i \to j) \cdot g = D(i \to j) \cdot g'.$$

As $D(i \to j)$ is monic, we have $g = g'$.

For the direction "⇐", we consider the morphisms

$$\big(\mathcal{C}(X, c_i) : \mathcal{C}(X, Di) \to \mathcal{C}(X, C)\big)_{i \in I}.$$

and need to prove that they are a colimit cocone for $\mathcal{C}(X, D-)$ under the conditions 1, 2, and that c_i is a colimit cocone for D. As $\mathcal{C}(X, -)$ preserves equality, we have commuting triangles $\mathcal{C}(X, c_i) = \mathcal{C}(X, c_j) \cdot \mathcal{C}(X, D(i \to j))$, i.e. the $\mathcal{C}(X, c_i)$ form a cocone. In Set, we know that

$$L := \coprod_{i \in I} \mathcal{C}(X, Di) /_{\sim} \quad \text{with} \quad [-]_i : \mathcal{C}(X, Di) \to L$$

form a colimit cocone for $\mathcal{C}(X, D-)$, where $\sim$ is the least equivalence relation with

$$X \xrightarrow{a} Di \ \sim \ X \xrightarrow{b} Dj \quad \text{if there exists } (i \to j) \text{ s.t. } \mathcal{C}(X, D(i \to j))(a) = b. \tag{4.1}$$

In the following, we construct a cocone isomorphism $u : \mathcal{C}(X, C) \to L$. For a given $f : X \to C$, consider the induced $g : X \to Di$ and define $u(f) := [g]_i$. For the following two reasons u is well defined:

- For some fixed i, $u(f)$ is uniquely defined, because $g : X \to Di$ is unique by condition 2.
- Assume $f = c_i \cdot g = c_j \cdot h$ for $h : X \to Dj$. As $(I, \leq)$ is directed, there is some upper bound b for i and j, with

  $$c_i = c_b \cdot D(i \to b) \text{ and } c_j = c_b \cdot D(j \to b).$$

  Consider the – not trivially commuting – diagram

  $$\begin{tikzcd}
  X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[drrr, bend right, "h"'] & & C & \\
  & Di \arrow[ur, "c_i"] \arrow[r, tail] & Db \arrow[u, "c_b"] & Dj \arrow[ul, "c_j"'] \arrow[l, tail]
  \end{tikzcd}$$

  in which all parts commute, except possibly for the bottom $D(j \to b) \cdot h \stackrel{?}{=} D(i \to b) \cdot g$. But $c_b \cdot D(j \to b) \cdot h = c_b \cdot D(i \to b) \cdot g$ and so by condition 2, $D(j \to b) \cdot h = D(i \to b) \cdot g$ and therefore

  $$[g]_i = [D(j \to b) \cdot g]_b = [D(j \to b) \cdot h]_b = [h]_j$$

  So $u(f)$ is independent from the choice of i.

Furthermore, $u : \mathcal{C}(X, C) \to L$ is a cocone morphism, i.e. for any $i \in I$ the diagram

$$\begin{tikzcd}
\mathcal{C}(X, C) \arrow[rr, "u"] & & L \\
& \mathcal{C}(X, Di) \arrow[ul, "{\mathcal{C}(X, c_i)}"] \arrow[ur, "{[-]_i}"'] &
\end{tikzcd}$$

commutes. That is because for $a : X \to Di$, $u(\mathcal{C}(X, c_i)(a)) = u(c_i \cdot a) = [a]_i$.

It remains to show that u is an isomorphism in Set. With the just proved equality we have

$$\mathrm{Im}(u) \supseteq \bigcup_{i \in I} \mathrm{Im}(u \cdot \mathcal{C}(X, c_i)) = \bigcup_{i \in I} \mathrm{Im}([-]_i) = L,$$

i.e. u is surjective. For injectivity consider $f, f' : X \to C$ with corresponding $f = c_i \cdot g$, $f' = c_j \cdot g'$, $g : X \to Di, g' : X \to Dj$ and with $u(f) = u(f')$. I.e. $[g]_i = [g']_j$. Let b an upper bound for i and j. So $[D(i \to b) \cdot g]_b = [D(j \to b) \cdot g']_b$, which means that there is some $\mathcal{C}(X, D(b \to p))$ with

$$\begin{aligned} &\mathcal{C}(X, D(b \to p))(D(i \to b) \cdot g) = \mathcal{C}(X, D(b \to p))(D(j \to b) \cdot g') \\ \Leftrightarrow\ &D(b \to p) \cdot D(i \to b) \cdot g = D(b \to p) \cdot D(j \to b) \cdot g'. \end{aligned}$$

But as $D(b \to p)$ is monic and as $\mathcal{C}(X, C)$ is the vertex of a cocone, we have

$$f = \mathcal{C}(X, c_i)(g) = \mathcal{C}(X, c_b)(D(i \to b) \cdot g) = \mathcal{C}(X, c_b)(D(j \to b) \cdot g') = \mathcal{C}(X, c_j)(g') = f'.$$

Therefore, u is injective and in total a colimit cocone isomorphism $\mathcal{C}(X, C) \cong L$. □

Proposition 4.8 ([AR94, Proposition 1.69]). *In an lfp category:*

- *each strong quotient of a finitely generated object is finitely generated.*
- *each finitely generated object is the strong quotient of a finitely presentable object.*

So, the two classes of objects coincide iff finitely presentable objects are closed under strong quotients.

Proposition 4.9. *Finitely generated objects are closed under finite colimits.*

Proof. Consider a finite diagram $D : \mathcal{D} \to \mathcal{B}$ with Di f.g. for all $i \in \mathbf{obj}\,\mathcal{D}$. Every Di is a strong quotient of a f.g. object, denote this quotient by the strong epi $q_i : B_i \twoheadrightarrow Di$.

$$\coprod_{i \in \mathbf{obj}\,\mathcal{D}} B_i \xrightarrow{\coprod_{i \in \mathbf{obj}\,\mathcal{D}} q_i} \coprod_{i \in \mathbf{obj}\,\mathcal{D}} Di \xrightarrow{[\mathsf{in}_i]_{i \in \mathbf{obj}\,\mathcal{D}}} \operatorname{colim} D$$

The coproduct of the quotients q_i is itself a strong epi, and by Proposition 3.13 the copair of the colimit injections in_i as well. By [AR94, Proposition 1.3], the finite coproduct of f.p. objects $\coprod B_i$ is f.p., and so its strong quotient $\operatorname{colim} D$ is finitely generated. □

Proposition 4.10 ([AR94, Proposition 1.61]). *Every lfp category has (strong epi,mono)-factorizations.*

Proposition 4.11 ([AR94, Proof I of Theorem 1.70]). *In an lfp category $\mathcal{B}$ there exists up to isomorphism only a set of finitely generated objects. Every object B of $\mathcal{B}$ is a directed colimit of all its f.g. subobjects, where both the connecting morphisms and the colimit injections are monos.*

Although this Proposition 4.11 is strictly stronger than the necessity direction of the equivalence in [AR94, Theorem 1.70], it still follows from the same proof.

4.2 Coalgebras

Let $\mathcal{B}$ be some category and let $F : \mathcal{B} \to \mathcal{B}$ denote an endofunctor on $\mathcal{B}$. An F-coalgebra is an object $X \in \mathcal{B}$, called carrier, together with a morphism

$$X \xrightarrow{x} FX,$$

called the structure of the coalgebra. An F-coalgebra homomorphism $h : (X, x : X \to FX) \to (Y, y : Y \to FY)$ is a morphism $h : X \to Y$ in $\mathcal{B}$ which preserves the coalgebra structure, i.e. for which the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{x} & FX \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ Y & \xrightarrow{y} & FY \end{array}$$

Definition 4.12. For a concrete functor F, the F-coalgebras together with their homomorphisms form a category, which is denoted by $\mathsf{Coalg}F$. If F is clear from the context, $\mathsf{Coalg}F$ is abbreviated to Coalg.

For a tutorial on coalgebras and results about them, see [JR97]. For example the terminal object in $\mathsf{Coalg}F$ – the final F-Coalgebra – has certain properties. In particular, by Lambek's Lemma, it is a fixpoint of F, which we will denote by $(\nu F, \tau : \nu F \to F\nu F)$. In general, a coalgebra $(X, x : X \to FX)$ is considered a fixpoint for F if x is an isomorphism, i.e. if x witnesses $X \cong FX$. It is a well-known and important result, that for a finitary functor on a complete category, the final coalgebra exists.

Example 4.13.
- For $FX = \mathbb{N} + X \times X$ on Set, the final F-coalgebra consists of possibly infinite binary trees with natural numbers in the leaves.
- On Set and for a fixed set C, coalgebras for the endofunctor $FX = C$, can be regarded as sets with a "coloring", i.e. sets in which each element has a "color" from C. F-coalgebra homomorphisms preserve the coloring, and the final F-coalgebra is $\mathrm{id}_C : C \to FC$ itself.
- For the powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, $\mathcal{P}$-coalgebras are directed graphs. The $\mathcal{P}$-coalgebra homomorphisms are the edge-reflecting graph homomorphisms. For $e : V \to \mathcal{P}V$, each vertex $v \in V$ is mapped to the set of its neighbours $e(v) \subseteq V$. But $\mathcal{P}$ has no final coalgebra, because by Cantor's theorem, there is no isomorphism $X \to \mathcal{P}X$ for any set X.

Many properties from the base category $\mathcal{B}$ lift to $\mathsf{Coalg}F$. For example, it is commonly known that colimits in $\mathsf{Coalg}F$ are created by the forgetful functor $U : \mathsf{Coalg}F \to \mathcal{B}$. More important for this work is, that monomorphisms, strong epimorphisms, and (strong epi, mono)-factorizations lift in a weak but still strong enough sense, as the following propositions show.

Lemma 4.14. *If $\mathcal{B}$ is lfp and F finitary, then forgetful functor $U : \mathsf{Coalg}F \to \mathcal{B}$ has a right adjoint.*

Proof. Combining Lemma 5.7 and Lemma 5.6 (i)→(iii) from [GLdMP01] gives that U has a right adjoint. □

Corollary 4.15. *If $\mathcal{B}$ is lfp and F finitary, $U : \mathsf{Coalg}F \to \mathcal{B}$ preserves both epimorphisms and strong epimorphisms.*

Proof. As U is a left adjoint, it preserves epis and by Proposition 3.17 also strong epis. □

Proposition 4.16. *If $m : (X, x) \to (Y, y)$ is a monomorphism $m : X \to Y$ in $\mathcal{B}$, it is also a monomorphism in $\mathsf{Coalg}F$.*

Proof. This is just straightforward: assume $f, g : (W, w) \to (X, x)$ with $m \cdot f = m \cdot g$ in $\mathsf{Coalg}F$. Then $m \cdot f = m \cdot g$ holds in $\mathcal{B}$ as well, so it is $f = g$ in $\mathcal{B}$ and in $\mathsf{Coalg}F$. □

Note that the converse direction does not necessarily hold:

Example 4.17 ([GS05, Example 3.5])**.** Define the Set-endofunctor $F := (-)^3_2$ on sets as

$$X^3_2 := \{(x_1, x_2, x_3) \in X^3 \mid x_1 = x_2 \text{ or } x_2 = x_3 \text{ or } x_3 = x_1\}$$

and on maps component-wise. It preserves monos, as (nearly) all injective maps on Set are split, except for the monos $m : 0 \rightarrowtail X$ which is indeed mapped to the mono $Fm : 0 = F0 \rightarrowtail FX$.

F is a quotient of the signature functor for the signature Σ consisting of three binary symbols; or equivalently, F is the quotient q of the polynomial $3 \times (-)^2$, with

$$q_X(k, x, y) = \begin{cases} (x, x, x) & \text{if } x = y \qquad (x, y, y) \quad \text{if } k = 0 \\ (y, x, y) & \text{if } k = 1 \qquad (y, y, x) \quad \text{if } k = 2, \end{cases}$$

see also [AM06, Examples 2.5(iii)]. So F as the quotient of a signature functor on Set is finitary by [AT90, Proposition 4.3]. Define the F-coalgebra structure on $B = \{0, 1\}$ as

$$b : x \mapsto (0, x, 1) \in \{0, 1\}^3_2.$$

Then one can follow that for every coalgebra $a : A \to FA$ there is at most one coalgebra homomorphism $h : (A, a) \to (B, b)$: consider some $p \in A$ with $a(p) = (x, y, z)$.

$$\begin{array}{llllll} \text{If } x = z & \Rightarrow & h(x) = h(z) & \Rightarrow & b \cdot h(p) = Fh \cdot a(p) = (h(x), h(y), h(x)) \notin \mathsf{Im}(b) \lightning \\ \text{If } x = y & \Rightarrow & h(x) = h(y) & \Rightarrow & Fh \cdot a(p) = (h(y), h(y), h(z)) = b \cdot h(y) \Rightarrow h(y) = 0 \\ \text{If } y = z & \Rightarrow & h(y) = h(z) & \Rightarrow & Fh \cdot a(p) = (h(x), h(y), h(y)) = b \cdot h(y) \Rightarrow h(y) = 1. \end{array}$$

So any morphism with domain (B, b) is a monomorphism in $\mathsf{Coalg}F$, e.g.

$$!_B : \ (B, b) \longrightarrow\!\!\!\!\rightarrow (1, 1 \to F1 = 1).$$

But $!_B$ is clearly not injective, i.e. not monic in Set.

Because of that example, we need to consider a stronger notion of monomorphism.

Definition 4.18 (mono-carried, strong epi-carried)**.** For some endofunctor $F : \mathcal{B} \to \mathcal{B}$, a coalgebra homomorphism $m : (A, a) \to (B, b)$ is called *mono-carried* if the morphism $m : A \to B$ is monic in $\mathcal{B}$. Furthermore, a homomorphism: $e : (A, a) \to (B, b)$ is called *strong epi-carried* if it is carried by a strong epimorphism. By a strong quotient of a coalgebra we understand (a quotient represented by) a strong-epi carried coalgebra homomorphism.

So the previous Corollary 4.15 just says that all strong epis in $\mathsf{Coalg}F$ are strong epi-carried.

Proposition 4.19. *Any strong epi-carried* $e : (A, a) \to (B, b)$ *is left orthogonal to any mono-carried* $m : (C, c) \to (D, d)$.

Proof. Take arbitrary $f : (A, a) \to (C, c)$ and $g : (B, b) \to (D, d)$ with $m \cdot f = g \cdot e$. The following diagram commutes: the inner square by assumption and each outer trapezoid because those are coalgebra homomorphisms.

$$\begin{tikzcd}
FA \arrow[rrr, "Fe"] \arrow[ddd, "Ff"'] & & & FB \arrow[ddd, "Fg"] \arrow[dddlll, bend right=60, "Fu"'] \\
& A \arrow[ul, "a"'] \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[ur, "b"] \arrow[d, "g"] \arrow[dl, dashed, "u"'] & \\
& C \arrow[dl, "c"'] \arrow[r, tail, "m"] & D \arrow[dr, "d"] & \\
FC \arrow[rrr, "Fm"] & & & FD
\end{tikzcd}$$

This induces the unique diagonal $u : B \to C$ with $u \cdot e = f$ and $m \cdot u = g$. Just from the fact that u is a diagonal, we also have $Fu \cdot Fe = Ff$, all small parts of the above diagram commute, and as e is epic, u is a coalgebra homomorphism. Any other suitable $\bar{u} : (B, b) \to (C, c)$ would be a suitable diagonal $\bar{u} : B \to C$, so $u : (B, b) \to (C, c)$ is the unique diagonal. $\square$

Proposition 4.20. *For an endofunctor $F : \mathcal{B} \to \mathcal{B}$ which preserves monos, a (strong epi,mono)-factorization lifts to a (strong epi-carried,mono-carried)-factorization in the category of F-coalgebras.*

Proof. For an F-coalgebra homomorphism $h : (X, x) \to (Y, y)$, take the (strong epi,mono)-factorization

$$h = (\ X \xrightarrow{e} \!\!\!\!\twoheadrightarrow I \rightarrowtail^{m} Y\ )$$

in $\mathcal{B}$. By diagonalization, I can be equipped with a coalgebra structure:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowleft"] & I \arrow[r, tail, "m"] \arrow[d, dashed, "\exists!"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "y"] \\ FX \arrow[r, "Fe"] & FI \arrow[r, tail, "Fm"] & FY \end{tikzcd}$$

□

4.3 From LFP to LFG Coalgebras

Assumption 4.21. In this section, assume that $\mathcal{B}$ is a locally finitely presentable category and $H : \mathcal{B} \to \mathcal{B}$ a finitary endofunctor preserving monos.

Definition 4.22 (lfp coalgebra, [Mil10, Definition 3.7])**.** $X \xrightarrow{x} HX$ is called an lfp coalgebra if

1. for all finitely presentable objects S and $f : S \to X$, there exists some finitely presentable P with coalgebra structure $p : P \to HP$ and a coalgebra morphism $h : (P, p) \to (X, x)$ and $f' : S \to P$ such that:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f'"'] & \circlearrowleft & X \\ & P \arrow[ur, "h"'] & \end{tikzcd}$$

2. The factorization is essentially unique: for another $f'' : S \to P$ with $h \cdot f'' = f$ there is some H-coalgebra (Q, q) with Q finitely presentable and coalgebra homomorphisms $(P, p) \xrightarrow{l} (Q, q) \xrightarrow{h'} (X, x)$ such that $l \cdot f' = l \cdot f''$ and $h = h' \cdot l$.

Let us adapt the definition of an lfp coalgebra to the notion of an lfg coalgebra.

Definition 4.23 (lfg coalgebra, bloated version)**.** $X \xrightarrow{x} HX$ is called an lfg coalgebra if

1. for all finitely generated objects $f : S \to X$, there exists some finitely generated P with coalgebra structure p and a coalgebra homomorphism $h : (P, p) \to (X, x)$ and $f' : S \to P$ such that:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f'"'] & \circlearrowleft & X \\ & P \arrow[ur, "h"'] & \end{tikzcd}$$

2. The factorization is essentially unique: for another $f'' : S \to P$ with $h \cdot f'' = f$ there is some H-coalgebra (Q, q) with Q finitely generated and coalgebra homomorphisms $(P, p) \xrightarrow{l} (Q, q) \xrightarrow{h'} (X, x)$ such that $l \cdot f' = l \cdot f''$ and $h = h' \cdot l$.

As finitely generated objects are closed under strong quotients, this definition can be simplified a lot. Observe that the second condition is implied by the first condition:

Proof of obsoleteness of uniqueness. Assume two such $f', f'' : S \to P$. Factor h according to Proposition 4.20 into

$$h = (\ (P,p) \xrightarrow{l} \!\!\!\!\rightarrow (Q,q) \rightarrowtail^{h'} (X,x)\ ).$$

Since we have $h \cdot f' = h \cdot f''$ and h' is a mono, we obtain $l \cdot f' = l \cdot f''$. As finitely generated objects are closed under strong quotients, Q is f.g. and all required conditions are met. $\square$

We can go even further to the following equivalent definition:

Definition 4.24 (lfg coalgebra, compact version)**.** $X \xrightarrow{x} HX$ is called an lfg coalgebra if for all finitely generated subobjects $f : S \rightarrowtail X$, there exists some finitely generated P with coalgebra structure p and a coalgebra morphism $h : (P,p) \to (X,x)$ and $f' : S \rightarrowtail P$ such that:

$$\begin{tikzcd}
S \arrow[rr, tail, "f"] \arrow[dr, tail, "f'"'] & \circlearrowleft & X \\
& P \arrow[ur, tail, "h"'] &
\end{tikzcd}$$

In other words: every finitely generated subobject of the carrier X is contained in the carrier of some finitely generated subcoalgebra of (X,x).

Proof of equivalence. $\Rightarrow$ Assume (X,x) has the properties according to Definition 4.23 and consider some subobject $f : S \rightarrowtail X$. We get a finitely generated coalgebra (P,p), a morphism $f' : S \to P$ and some $h : (P,p) \to (X,x)$. According to Proposition 4.20, h factors into

$$h = (\ (P,p) \xrightarrow{h_e} \!\!\!\!\rightarrow (P',p') \rightarrowtail^{h_m} (X,x)\ )$$

with $f = h \cdot f' = h_m \cdot (h_e \cdot f')$. The intermediate coalgebra (P',p') is the desired subcoalgebra of (X,x) and, as f is a monomorphism, $(h_e \cdot f')$ is a monomorphism as well. Note that P' is finitely generated as it is a strong quotient of the finitely generated P.

$\Leftarrow$ Now assume (X,x) has the property from Definition 4.24 and consider some object $f : S \to X$. Factor f into $f_e : S \twoheadrightarrow I$ and $f_m : I \rightarrowtail X$. I is a strong quotient of the f.g. X and thus f.g. itself, so we can apply it to the lfg-property to get (P,p), $h : (P,p) \rightarrowtail (X,x)$, and $f' : I \rightarrowtail P$. $\square$

Proposition 4.25. *Every lfp coalgebra is lfg.*

Proof. Given an lfp coalgebra (X,x) and a finitely generated subobject $f : S \rightarrowtail X$, S is the strong quotient of some finitely presentable object $q : S' \twoheadrightarrow S$. By the morphism $f \cdot q : S' \to X$ and the first condition of Definition 4.22, we get a coalgebra (P,p) with finitely presentable carrier and a morphism $h : (P,p) \to (X,x)$. This h (strong epi-carried, mono-carried)-factors into h_e and h_m through some (P',p'). As P' is the quotient of P, P' is finitely generated. The following commuting diagram summarizes the steps up to now:

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, two heads, "q"'] \arrow[dr, "f \cdot q"] & P \arrow[d, "h"] \arrow[dr, two heads, "h_e"] & \\
S \arrow[r, tail, "f"'] & X & P' \arrow[l, tail, "h_m"]
\end{tikzcd}$$

The strong epimorphism q together with the mono h_m induces a unique diagonal $f' : S \to P'$, with $h_m \cdot f' = f$ in particular. As h_m and f are monomorphisms, f' is as well. As H preserves monos, the intermediate P' carries a coalgebra structure such that h_m and h_e are coalgebra homomorphisms according to Proposition 4.20. $\square$

Corollary 4.26. *Any coalgebra (X, x) with finitely generated carrier X is lfg.*

Definition 4.27. By $\mathsf{Coalg}_{\mathrm{lfg}}H$ denote the full subcategory of lfg coalgebras; also denote by $\mathsf{Coalg}_{\mathrm{fg}}H$ the full subcategory of $\mathsf{Coalg}H$ and $\mathsf{Coalg}_{\mathrm{lfg}}H$ consisting of coalgebras with finitely generated carrier.

Proposition 4.28. *$\mathsf{Coalg}_{\mathrm{fg}}H$ is closed under finite colimits.*

Proof. Colimits in $\mathsf{Coalg}H$ are created by the forgetful functor $\mathsf{Coalg}H \to \mathcal{B}$. Finitely generated objects are closed under finite colimits, see Proposition 4.9. So the carrier of the colimit is f.g., and the colimit is in $\mathsf{Coalg}_{\mathrm{fg}}H$. $\square$

Remark 4.29. By a *directed union* of coalgebras we mean the colimit of a directed diagram in $\mathsf{Coalg}H$ where the connecting homomorphisms are mono-carried.

Observation 4.30. Every directed union of coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}H$ is an lfg coalgebra.

Proof. Let $D : (I, \leq) \to \mathsf{Coalg}H, (D_i, d_i) := Di$ be a directed diagram of coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}H$ and of mono-carried morphisms. Name the colimit cocone $c_i : (D_i, d_i) \to (A, a)$ in $\mathsf{Coalg}H$. To check Definition 4.23, let S be a finitely generated object with $f : S \to A$ in $\mathcal{B}$. As colimits in $\mathsf{Coalg}H$ are created by the forgetful functor $U : \mathsf{Coalg}H \to \mathcal{B}$, and because $U \cdot D$ is a directed diagram of monos, Theorem 4.7 gives us some factorization:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f'"'] & & U(A,a) = A \\ & UDi = D_i \arrow[ur, "Uc_i"'] & \end{tikzcd}$$

Note that because U creates the colimits, we know that the colimit injection for UDi in $\mathcal{B}$ is precisely Uc_i. $\square$

We need a few lemmas before we can observe, that also every filtered diagram of coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}H$ is an lfg coalgebra.

Lemma 4.31. *For a directed diagram $D : \mathcal{D} \to \mathcal{B}$ of subobjects $m_i : C_i \rightarrowtail C$ of C, the colimit $(d_i : C_i \to \operatorname{colim} D)_{i \in \mathcal{D}}$ is obtained by taking the (strong epi,mono)-factorization of $\coprod C_i \xrightarrow{[m_i]} C$.*

Proof. At first, the $(m_i)_{i \in \mathcal{D}}$ form a cocone, so we have a unique $m : \operatorname{colim} D \to C$ with $m \cdot d_i = m_i$, and d_i is monic. As $\mathcal{B}$ is lfp and both d_i and m_i are monic, [AR94, Proposition 1.62 (ii)] gives us that m is monic, too. By Proposition 3.13, $[d_i] : \coprod C_i \to \operatorname{colim} D$ is a strong epi and therefore we have the factorization:

$$\begin{tikzcd} \coprod C_i \arrow[rr, "{[m_i]}"] \arrow[dr, two heads, "{[d_i]}"'] & & C \\ & \operatorname{colim} D \arrow[ur, tail, "m"'] & \end{tikzcd}$$

$\square$

Lemma 4.32. *Images of colimits in $\mathsf{Coalg}H$ are directed unions of images. More precisely, for a diagram $D : \mathcal{D} \to \mathsf{Coalg}H$, given a colimit cocone $(c_i : Di \to C)_{i \in \mathcal{D}}$ and a morphism $f : C \to B$, define A_i as $\mathsf{Im}(f \cdot c_i)$. Then $\mathsf{Im}(f)$ is the directed union of the A_i together with the induced monomorphisms.*

Proof. As colimits in $\mathsf{Coalg}H$ are created by the forgetful functor $U : \mathsf{Coalg}H \to \mathcal{B}$, we consider only the objects first. Take the (strong epi-carried,mono-carried)-factorizations $f \cdot c_i = m_i \cdot e_i$ for each $i \in \mathcal{D}$, and $f = m \cdot e$. Consider the commuting

$$\begin{tikzcd}
Di \arrow[rr, two heads, "e_i"] \arrow[d, "c_i"'] & & A_i \arrow[d, tail, "m_i"] \arrow[dl, dashed, "d_i"'] \\
C \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & \mathsf{Im}(f) \arrow[r, tail, "m"] & B
\end{tikzcd} \tag{4.2}$$

where d_i is induced by the strong epi e_i. Notice that by $m \cdot d_i = m_i$, d_i is a mono as well. For any morphism $g : Di \to Dj$ we get a mono in $\bar{g} : A_i \rightarrowtail A_j$ by the strong epi e_i:

$$\begin{tikzcd}
Di \arrow[r, two heads, "e_i"] \arrow[d, "g"'] & A_i \arrow[d, tail, dashed, "\bar{g}"'] \arrow[dr, tail, "d_i"] & \\
Dj \arrow[r, two heads, "e_j"] & A_j \arrow[r, tail, "d_j"'] & \mathsf{Im}(f)
\end{tikzcd}$$

By $d_j \cdot \bar{g} = d_i$, we know that $\bar{g}$ is a mono as well. The d_i also ensure that between each pair of objects A_i, A_j there is at most one morphism. With this relation to the D_i, we also inherit the existence of upper bounds in A_i, which can be summarized in: the A_i form a directed diagram of monos in $\mathcal{B}$, i.e. a directed union in $\mathsf{Coalg}H$.

To see that $\mathsf{Im}(f)$ is indeed its colimit, consider

$$\begin{tikzcd}
\coprod_i Di \arrow[r, two heads, "{[c_i]}"] \arrow[d, two heads, "\coprod e_i"'] & C \arrow[d, two heads, "e"] \\
\coprod A_i \arrow[r, "{[d_i]}"'] & \mathsf{Im}(f)
\end{tikzcd}$$

which commutes, because (4.2) did for every $i \in \mathcal{D}$. By Proposition 3.13, $[c_i]$ is a strong epi and so $e \cdot [c_i]$ and $[d_i] \cdot \coprod e_i$ as well. Then by Proposition 3.12, $[d_i]$ is a strong epi and $[m_i]$ factors into m and $[d_i]$, and by Lemma 4.31 $\mathsf{Im}(f)$, is the colimit.

$$\begin{tikzcd}
\coprod A_i \arrow[r, two heads, "{[d_i]}"] \arrow[rr, bend left, "{[m_i]}"] & \mathsf{Im}(f) \arrow[r, tail, "m"] & B
\end{tikzcd}$$

□

Applying Lemma 4.32 to a strong epimorphism $f : C \twoheadrightarrow B$ directly gives:

Observation 4.33. Every filtered colimit of coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}H$ is lfg, using that H preserves monos. Indeed, let $c_i : (X_i, x_i) \to (X, x)$ be a colimit cocone of a filtered diagram with (X_i, x_i) from $\mathsf{Coalg}_{\mathrm{fg}}H$. Take the (strong epi,mono)-factorizations

$$c_i \equiv (\ X_i \xrightarrow{e_i} \!\!\!\!\rightarrow T_i \overset{m_i}{\rightarrowtail} X\ )$$

to get the subcoalgebras (T_i, t_i) of (X, x). By Lemma 4.32 with $f = \mathrm{id}_X : X \to X$, $\mathsf{Im}(f) = X$ is the directed union of the T_i. These T_i are in $\mathsf{Coalg}_{\mathrm{fg}}H$ since strong quotients of finitely generated objects are finitely generated. This diagram of the T_i is a directed union with colimit (X, x), both in $\mathcal{B}$ and in $\mathsf{Coalg}H$, so according to Observation 4.30, (X, x) is lfg.

Proposition 4.34. *Every lfg coalgebra is a directed union of its subcoalgebras from* $\mathsf{Coalg}_{\mathrm{fg}}H$*.*

Proof. Let (X, x) be an lfg coalgebra. Consider the diagram of all its subcoalgebras in $\mathsf{Coalg}_{\mathrm{fg}}H$. We will prove that (X, x) is the directed union of this diagram.

Recall from Proposition 4.11 that X is the colimit of the diagram of all its finitely generated subobjects. Let us verify that the above diagram of subcoalgebras is a cofinal subdiagram. This follows directly from the fact that (X, x) is an lfg coalgebra: for every subobject $S \rightarrowtail X$, S f.g., we have a subcoalgebra of (X, x) in $\mathsf{Coalg}_{\mathrm{fg}}H$ containing S. □

Corollary 4.35. *The lfg coalgebras are precisely the directed unions of coalgebras from* $\mathsf{Coalg}_{\mathrm{fg}}H$*.*

From that and from Observation 4.33, it follows:

Corollary 4.36. *The final lfg coalgebra is the colimit of the filtered diagram of all coalgebras in* $\mathsf{Coalg}_{\mathrm{fg}}H$*.*

Similar to finitely generated objects, lfg coalgebras are closed under quotients:

Proposition 4.37. *Lfg coalgebras are closed under strong quotients.*

Proof. Consider some strong quotient $q : (X, x) \to (Y, y)$ where (X, x) is lfg. As (X, x) is the directed colimit of its subcoalgebras with f.g. carrier, we have that (Y, y) – the codomain of the strong epi-carried q – is the union of the images of these subcoalgebras by Lemma 4.32. The images themselves have a finitely generated carrier – more precisely the factorization in $\mathsf{Coalg}H$ exist because H preserves monos, by Proposition 4.20. So (Y, y) is the union of these images and thus is lfg, by Observation 4.30. □

Proposition 4.38. *Lfg coalgebras are closed under finite coproducts. More precisely, for the coproduct of two lfg coalgebras (X, x) and (Y, y), the subcoalgebra, which is induced by a subobject and the lfg property, is a coproduct of f.g. carried subcoalgebras of (X, x) and (Y, y).*

Proof. This directly follows from that fact, that both (X, x) and (Y, y) are filtered colimits of coalgebras with f.g. carrier. As colimits commute with colimits, we know that $X + Y \xrightarrow{x+y} HX + HY \xrightarrow{\mathsf{can}} H(X + Y)$ is the filtered colimit of the coproducts of f.g. carried subcoalgebras of (X, x) and (Y, y). Those coproducts are still f.g. carried, by Proposition 4.9, and so their filtered colimit $X + Y$ is lfg. □

4.4 Construction of the Locally Finite Fixpoint

In this section, we have the same assumptions as before, Assumption 4.21. We already know from Corollary 4.36, that the final lfg coalgebra is the colimit of $\mathsf{Coalg}_{\mathrm{fg}}H$. Moreover, we have a weaker criterion that ensures the finality of an lfg coalgebra.

Proposition 4.39. *An lfg coalgebra $(\mathscr{L}, \ell : \mathscr{L} \to H\mathscr{L})$ is final for all lfg coalgebras iff for all finitely generated $(X, x : X \to HX)$ there exists an unique coalgebra homomorphism $(X, x) \to (\mathscr{L}, \ell)$.*

The proof can be adapted from the analogous result [Mil10, Theorem 3.14] as follows:

Proof. The direction from left to right is clear, as $\mathsf{Coalg}_{\mathrm{fg}} \subseteq \mathsf{Coalg}_{\mathrm{lfg}}$. For the other one, let (S, s) be some lfg coalgebra. By Proposition 4.34, it is the directed union of all its subcoalgebras with finitely generated carrier. For each subcoalgebra $\mathrm{in}_p : (P, p) \to (S, s)$, there is a unique homomorphism into $\mathscr{L}$, let us call it $p^\dagger : (P, p) \to (\mathscr{L}, \ell)$. By the uniqueness of $p^\dagger$ it follows that $\mathscr{L}$ together with the $p^\dagger$ is a cocone. Hence there is a unique morphism $\exists! u : (S, s) \to (\mathscr{L}, \ell)$ with $u \cdot \mathrm{in}_p = p^\dagger$ for each appropriate (P, p). For any other morphism $\bar{u} : (S, s) \to (\mathscr{L}, \ell)$ the equation $\bar{u} \cdot \mathrm{in}_p = p^\dagger$ must hold as well, because $p^\dagger$ is unique. As the in_p are jointly epic, one gets $\bar{u} = u$. $\square$

In the remainder of the section, we will see that the final lfg coalgebra is indeed a fixpoint of H.

Definition 4.40. Any functor $H : \mathcal{B} \to \mathcal{B}$ lifts to a functor $\hat{H} : \mathsf{Coalg}H \to \mathsf{Coalg}H$ by

$$\hat{H}(C \xrightarrow{c} HC) = (HC \xrightarrow{Hc} HHC) \text{ and } \hat{H}(h : C \to D) = Hh$$

Lemma 4.41. *The lifted $\hat{H}$ acts just like H, namely:*

(i) If H preserves monos, then $\hat{H}$ preserves mono-carried homomorphisms.

(ii) $U\hat{H} = HU$, where $U : \mathsf{Coalg}H \to \mathcal{B}$ denotes the forgetful functor.

(iii) If H is finitary, then so $\hat{H}$ is.

Proof. The first two items are trivial. For (iii), consider some directed diagram $D : \mathcal{D} \to \mathsf{Coalg}H$. Recall that the forgetful functor U creates colimits, so we have

$$U\hat{H} \operatorname{colim} D \cong HU \operatorname{colim} D \cong H \operatorname{colim} UD \overset{(*)}{\cong} \operatorname{colim} HUD \cong \operatorname{colim} U\hat{H}D \cong U \operatorname{colim} \hat{H}D,$$

where $(*)$ holds because H is finitary. Again, as U creates colimits we have that a unique coalgebra structure is induced on $\operatorname{colim} U\hat{H}D$ and thus we also have that $\hat{H} \operatorname{colim} D \cong \operatorname{colim} \hat{H}D$. $\square$

Lemma 4.42. *For any lfg coalgebra $C \xrightarrow{c} HC$, the coalgebra $HC \xrightarrow{Hc} HHC = \hat{H}(C, c)$ is lfg.*

Proof. To check Definition 4.23, assume a finitely generated S from $\mathcal{B}$ and $f : S \to HC$. As $\mathcal{B}$ is lfp we have that HC is the colimit of its f.g. subobjects, i.e. a diagram of monos

$$D : \mathcal{D} \to \mathcal{B}, \quad \operatorname{colim} D = HC.$$

By Theorem 4.7, $f : S \to HC$ factors through some subobject $\mathsf{in}_q : Q \rightarrowtail HC$, $f = \mathsf{in}_q \cdot f'$. On the other hand, (C, c) is lfg, so it is the directed union of its subcoalgebras with f.g. carrier, or formally we have a diagram

$$E : \mathcal{E} \to \mathsf{Coalg}_{\mathrm{fg}}H, \text{ with } \operatorname{colim} E = (C, c).$$

By Lemma 4.41, $\hat{H}E$ is a directed diagram where the connecting morphisms are mono-carried and its colimit is

$$\operatorname{colim} \hat{H}E \cong \hat{H} \operatorname{colim} E \cong (HC, Hc : HC \to HHC).$$

Again, $U\hat{H}E$ is a directed diagram of monos, thus the morphism $\mathsf{in}_q : Q \to HC = \operatorname{colim} U\hat{H}E$ factors through some $U\hat{H}(P, p)$ via $\mathsf{in}_p : (P, p) \rightarrowtail (C, c)$, $H\mathsf{in}_p \cdot q = \mathsf{in}_q$, where $(P, p) \in \mathsf{Coalg}_{\mathrm{fg}}H$.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "f'"'] & HC \arrow[r, "Hc"] & HHC \\
Q \arrow[r, "q"'] \arrow[ur, "\mathsf{in}_q"] & HP \arrow[r, "Hp"'] \arrow[u, "H\mathsf{in}_p"'] & HHP \arrow[u, "HH\mathsf{in}_p"']
\end{tikzcd}$$

This commutes because in_p is a coalgebra homomorphism. Now we can construct a coalgebra with f.g. carrier

$$Q + P \xrightarrow{[q,p]} HP \xrightarrow{H\mathsf{inr}} H(Q+P)$$

and a coalgebra homomorphism $Q + P \to HC$. For the homomorphism consider

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[dd, "f'"'] & HC \arrow[rr, "Hc"] & & HHC \\
& HP \arrow[u, "H\mathsf{in}_p"'] \arrow[rr, "Hp"] \arrow[dr, equal] & & HHP \arrow[u, "HH\mathsf{in}_p"'] \\
Q \arrow[r, "\mathsf{inl}"] \arrow[uur, "\mathsf{in}_q"] \arrow[ur, "q"'] & Q + P \arrow[u, "{[q,p]}"] \arrow[r, "{[q,p]}"] & HP \arrow[r, "H\mathsf{inr}"] & H(Q+P) \arrow[u, "{H[q,p]}"'] \arrow[uu, bend right=60, "{H(H\mathsf{in}_p \cdot [q,p])}"']
\end{tikzcd}$$

in which every part trivially commutes. Hence $H\mathsf{in}_p \cdot [q,p]$ is the desired homomorphism. □

Using the previous lemma, we now prove a version of Lambek's lemma [Lam68, Lemma 2.2] for lfg coalgebras. In fact, by virtue of Lemma 4.42, the proof is identical to the one for ordinary final coalgebras.

Theorem 4.43. *For the final lfg coalgebra $\ell : \mathscr{L} \to H\mathscr{L}$, ℓ is an isomorphism.*

Proof. By Lemma 4.42, the H-coalgebra $H\ell : H\mathscr{L} \to HH\mathscr{L}$ is lfg and therefore induces a unique coalgebra homomorphism $h : (H\mathscr{L}, H\ell) \to (\mathscr{L}, \ell)$.

$$\begin{tikzcd}
\mathscr{L} \arrow[r, "\ell"] \arrow[d, "\ell"'] & H\mathscr{L} \arrow[d, "H\ell"] \\
H\mathscr{L} \arrow[r, "H\ell"] \arrow[d, "h"'] & HH\mathscr{L} \arrow[d, "Hh"] \\
\mathscr{L} \arrow[r, "\ell"] & H\mathscr{L}
\end{tikzcd}$$

The above square obviously commutes, so $h \cdot \ell$ is a coalgebra homomorphism $(\mathscr{L}, \ell) \to (\mathscr{L}, \ell)$. As $\mathrm{id}_{\mathscr{L}} : (\mathscr{L}, \ell) \to (\mathscr{L}, \ell)$ is another such coalgebra homomorphism, the finality of $\mathscr{L}$ gives us $h \cdot \ell = \mathrm{id}_{\mathscr{L}}$. Then, the bottom square gives us

$$\ell \cdot h = Hh \cdot H\ell = H(h \cdot \ell) = H\mathrm{id}_{\mathscr{L}} = \mathrm{id}_{H\mathscr{L}}.$$

So ℓ is an isomorphism with $h = \ell^{-1}$. □

Definition 4.44. From now on, we will call the final lfg H-coalgebra the *locally finite fixpoint* (LFF) of H, denoted by $\ell : \vartheta H \to H\vartheta H$.

Theorem 4.45. *The locally finite fixpoint is a subcoalgebra of the final coalgebra.*

Proof. Consider the unique coalgebra homomorphism $h : (\vartheta H, \ell) \to (\nu H, \tau)$ into the final coalgebra and take its factorization, according to Proposition 4.20:

$$\begin{tikzcd}
(\vartheta H, \ell) \arrow[r, two heads, "e"] \arrow[rr, bend left, "h"] & (J, j) \arrow[r, tail, "m"] \arrow[l, dashed, bend left, "i"] & (\nu H, \tau)
\end{tikzcd}$$

As by Proposition 4.37, lfg coalgebras are closed under strong quotients, (J, j) is lfg and thus induces a unique homomorphism i. As $\mathrm{id}_{\vartheta H}$ is the unique homomorphism $(\vartheta H, \ell) \to (\vartheta H, \ell)$, we have $i \cdot e = \mathrm{id}_{\vartheta H}$. Moreover, $e \cdot i \cdot e = e$, and as e is epic, we have $i \cdot e = \mathrm{id}_J$ and $(\vartheta H, \ell) \cong (J, j)$. $\square$

4.5 Relation to the Rational Fixpoint

For a lfp category $\mathcal{B}$ and an finitary endofunctor H, we know that the final H-coalgebra

$$\tau : \nu H \to H \nu H$$

exists, and by Lambek's Lemma that it is a fixpoint. The finality gives us that for *every* coalgebra $c : C \to HC$, there is a unique coalgebra homomorphism from C to νH. In many applications, we are interested in coalgebras with a finite carrier. So more recently, it has been found out that there is a final lfp coalgebra

$$r : \varrho H \to H \varrho H$$

which has this finality property only for lfp coalgebras [Mil10]. Furthermore, ϱH is a fixpoint of H, i.e. r is an isomorphism [AMV06]. As νH is final, we have a unique morphism $\varrho H \to \nu H$. But ϱH is not necessarily a subcoalgebra of νH, see [BMS13, Example 3.15]. But we will see in this section, that the missing piece in between is indeed our locally finite fixpoint.

Let us first look at the general setting for fixpoints of functors. Observe that any isomorphism $i : I \xrightarrow{\cong} HI$ can be read both as an algebra and as a coalgebra, and for another iso $j : J \to HJ$ a morphism $h : I \to J$ is an algebra homomorphism $(I, i^{-1}) \to (J, j^{-1})$ iff it is a coalgebra homomorphism $(I, i) \to (J, j)$.

For some conditions, such a (co)algebra homomorphism gives us another fixed point of H.

Lemma 4.46. *Let $H : \mathcal{B} \to \mathcal{B}$, for $\mathcal{B}$ an lfp category, A and C be two fixpoints and $f : A \to C$ a (co)algebra homomorphism which (strong epi,mono)-factors into $f = m \cdot e$ through B. Then the following conditions are equivalent:*

(i) The morphism He is a strong epi and Hm a mono.

(ii) There is some isomorphism $B \cong HB$ making e and m (co)algebra homomorphisms.

Proof. For the direction (i)$\Rightarrow$(ii) consider:

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] \arrow[d, tail, two heads, "i_A"', "\cong"] & B \arrow[r, tail, "m"] \arrow[d, dashed, shift left, "i_B"] & C \arrow[d, tail, two heads, "i_C"', "\cong"] \\
HA \arrow[r, two heads, "He"] & HB \arrow[r, tail, "Hm"] \arrow[u, dashed, shift left, "j_B"] & HC
\end{tikzcd}$$

As e is a strong epi and $i_C^{-1} \cdot Hm$ is a mono, we get a unique i_B which lets both smaller squares commute. Analogously, $He \cdot i_A$ is a strong epi and m a mono, so we get a unique j_B making both smaller squares commute. This implies that $j_B \cdot i_B$ and $i_B \cdot j_B$ are the unique diagonals in the squares

$$
\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "e \cdot i_A^{-1} \cdot i_A"'] & B \arrow[d, "i_C^{-1} \cdot i_C \cdot m"] \\
B \arrow[r, tail, "m"'] & C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
HA \arrow[r, two heads, "He"] \arrow[d, "He \cdot i_A^{-1} \cdot i_A"'] & HB \arrow[d, "i_C^{-1} \cdot i_C \cdot Hm"] \\
HB \arrow[r, tail, "Hm"'] & HC
\end{tikzcd}
$$

respectively. As the unique diagonals are id_B and id_{HB} it follows that $j_B = i_B^{-1}$, hence B is a fixpoint of H.

For the other direction (ii)⇒(i) we have $i_B^{-1} \cdot He \cdot i_A = e$ and $i_C^{-1} \cdot Hm \cdot i_B = m$, and as i_A, i_B, i_C are isos, we have (i) directly. □

In the following let us see, how the intermediate object between the rational fixed point and the final coalgebra looks like. Therefore, we need some additional assumptions, which will be checked for a couple of examples later.

Definition 4.47. A strong epi projective is an object X such that for every strong epi $e : A \twoheadrightarrow B$ and every $f : X \to B$ there is some (not necessarily unique) $f' : X \to A$ with $f = e \cdot f'$.

Assumption 4.48. In addition to Assumption 4.21, assume that in the base category $\mathcal{B}$, every finitely presentable object is a strong quotient of a finitely presentable strong epi projective object and that the endofunctor H also preserves strong epis.

Though this assumption sounds very strong, it is met in many situations. The condition that every object is the strong quotient of a strong epi projective often is phrased as *having enough strong epi projectives*. See also [Bor94] for details and for a list of examples. For our concerns, the following categories are of interest.

Example 4.49.

- In categories in which all (strong) epis are split, every object is projective, e.g. in Set or Vec_K.
- In the category of finitary endofunctors $\mathsf{Fun_f}(\mathsf{Set})$, all polynomial functors are projective. The finitely presentable functors are quotients of polynomial functors H_Σ, where Σ is a finite signature.
- In the Eilenberg-Moore category Set^T for a finitary monad T, strong epis are surjective T-algebra homomorphisms. In Set^T, every free algebra TX is projective: for $e : A \twoheadrightarrow B$ and $f : TX \to B$, take the weak inverse $s : B \to A$ from Set with $e \cdot s = \mathrm{id}_B$. Then for $f_0 X \to B$, the composition $s \cdot f_0$ fulfills

  $$e \cdot (s \cdot f_0) = f_0,$$

  i.e. $\overline{s \cdot f_0} : VX \to A$ is the desired homomorphism.

  Every finitely generated object of Set^T is a strong quotient of some free algebra TX with X finite. For more details and the exact definitions of this example, see Section 5.1 later.

The general result that finitely generated objects are quotients of finitely presentable objects lifts to coalgebras:

Proposition 4.50. *Under Assumption 4.48, every coalgebra in $\mathsf{Coalg_{fg}}H$ is a strong quotient of a coalgebra with finitely presentable carrier.*

Proof. Take an coalgebra (X, x) with finitely generated carrier, which is the strong quotient of some f.p. object X' via $q : X' \twoheadrightarrow X$. By assumption, X' is the strong quotient of a projective f.p. object X'' via $q' : X'' \to X'$. As H preserves strong epis, the projectivity of X'' induces the coalgebra structure x'':

$$\begin{tikzcd}
X'' \arrow[r, dashed, "x''"] \arrow[d, two heads, "q'"'] & HX'' \arrow[d, two heads, "Hq'"] \\
X' \arrow[d, two heads, "q"'] & HX' \arrow[d, two heads, "Hq"] \\
X \arrow[r, "x"] & HX
\end{tikzcd}$$

□

Proposition 4.51. *Under Assumption 4.48, for $A := \varrho H$ and $C := \nu H$ in Lemma 4.46, B is the final lfg coalgebra.*

Proof. ϱH is the final lfp coalgebra, so it is in particular lfp and thus lfg. By Proposition 4.37, its strong quotient B then is lfg as well.

Let us prove that B has the sufficient finality property for coalgebras with finitely generated carrier, as stated in Proposition 4.39. So take $(X, x : X \to HX)$ with finitely generated X. By Proposition 4.50, (X, x) is the strong quotient of some $(P, p : P \to HP)$ with P finitely presentable. Now look at the final picture in $\mathsf{Coalg}H$, before building it step by step:

$$\begin{tikzcd}
(P,p) \arrow[r, two heads, "q"] \arrow[d, dashed, "\exists!\, p^\dagger"] & (X,x) \arrow[ddl, dashed, "\exists!\, u"] \arrow[dd, dashed, "\exists!\, x^\dagger"] \\
(\varrho H, r) \arrow[d, two heads, "e"] & \\
(B,\beta) \arrow[r, tail, "m"'] & (\nu H, \tau)
\end{tikzcd}$$

Now, let us see, how this is built: the finitely presentable (P, p) induces a unique coalgebra homomorphism into the rational fixpoint ϱH. Of course, (X, x) induces a unique morphism into the final coalgebra νH. As νH is final, the square necessarily commutes. As q is a strong epi and m a mono, there is a unique coalgebra morphism u from (X, x) to (B, β).

By now, u is only unique with respect to the commutativity of the two triangles of the diagram. The remaining question is, whether it is the unique coalgebra morphism $(X, x) \to (B, \beta)$. Consider another $\bar{u} : (X, x) \to (B, \beta)$. As νH is final, $\bar{u}$ lets the bottom right triangle commute. In other words: $m \cdot u = m \cdot \bar{u}$. As m is a mono, it follows that $u = \bar{u}$. □

Corollary 4.52. *The locally finite fixpoint is a quotient of the rational fixpoint, or more precisely is its image in the final coalgebra.*

4.6 Relation to Initial Iterative Algebras

Iterative algebras provide a notion of unique solutions for recursive equations. For the finitely presentable case, it was shown in [AMV06] how iterative algebras are connected to lfp coalgebras. In more detail, it turns out that the initial iterative algebra is the final lfp coalgebra.

In the following, this result is adapted to the finitely generated case, beginning with the formal definition of *recursive equations*.

Assumption 4.53. In this section, assume the lfp category $\mathcal{B}$ with a mono-preserving endofunctor $H : \mathcal{B} \to \mathcal{B}$.

It follows that we have the locally finite fixpoint $(\vartheta H, \ell)$.

Definition 4.54. An *equation morphism* e in an object A is a morphism in $\mathcal{B}$

$$X \to HX + A,$$

where X is a finitely generated object. If A is the carrier of an algebra $\alpha : HA \to A$, we call the $\mathcal{B}$-morphism $e^\dagger : X \to A$ a solution of e if

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\
HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[\alpha, A]}"']
\end{tikzcd}$$

commutes. An H-algebra A is called *fg-iterative* if every equation morphism has a unique solution in A.

Example 4.55 ([Mil05, Example 2.5 (iii)])**.** Final coalgebras are fg-iterative algebras. They even induce unique solutions of equation morphisms whose carrier is not necessarily finitely generated.

As a notion of homomorphism between fg-iterative algebras, we are interested in homomorphisms which preserve the solutions as follows.

Definition 4.56. For fg-iterative algebras A and C, an equation morphism $e : X \to HX + A$ and a morphism $h : A \to C$ of $\mathcal{B}$ define an equation morphism in C as follows.

$$h \bullet e \equiv \left(X \xrightarrow{e} HX + A \xrightarrow{HX + h} HX + C \right)$$

Then we say that h preserves the solution $e^\dagger$ of e if

$$\begin{tikzcd}
& X \arrow[dl, "e^\dagger"'] \arrow[dr, "(h \bullet e)^\dagger"] & \\
A \arrow[rr, "h"'] & & C
\end{tikzcd}$$

commutes. The morphism h is called *solution preserving* if it preserves the solution of any equation morphism e.

The property of a morphism $h : A \to C$ preserving solutions coincides with the one of being an algebra homomorphism. By [AMV06, Prop 2.18, Remark 2.19], algebra homomorphisms always preserve solutions. For the other direction we need to exploit that the carrier of the equation morphism is finitely generated.

Proposition 4.57. *Every solution preserving morphism $h : A \to C$ between fg-iterative algebras is an algebra homomorphism.*

Proof. Assume $h : A \to C$ preserves solutions and assume that $(A, \alpha : HA \to A)$ and $(C, \gamma : HC \to C)$ are fg-iterative algebras. Consider the following diagram for an arbitrary equation morphism $e : X \to HX + A$:

$$\begin{tikzcd}
X \arrow[dd, "e"'] \arrow[dr, "e^\dagger"] \arrow[drr, bend left, "(h \bullet e)^\dagger"] \arrow[ddd, bend right=90, "h \bullet e"'] \arrow[dr, phantom, bend left, "\text{(i)} \circlearrowleft"] & & \\
\arrow[r, phantom, "\text{(ii)} \circlearrowleft"] \arrow[d, phantom, "\text{(iii)} \circlearrowleft"'] & A \arrow[r, "h"] \arrow[r, phantom, shift right=6, "\text{(v)}"] & C \\
HX + A \arrow[r, "He^\dagger + A"] \arrow[d, "HX + h"] & HA + A \arrow[u, "{[\alpha, A]}"] \arrow[r, "Hh + h"] \arrow[d, phantom, "\text{(iv)} \circlearrowleft"] & HC + C \arrow[u, "{[\gamma, C]}"'] \\
HX + C \arrow[urr, bend right, "H(h \bullet e)^\dagger + C"'] & &
\end{tikzcd}$$

In the following we show that the parts (i)–(iv) commute. The outside commutes because $(h \bullet e)^\dagger$ is the solution of $h \bullet e$ in C.

(i) This just says that h is solution preserving.

(ii) $e^\dagger$ is a solution for e in A, so (ii) commutes.

(iii) Holds by definition of $h \bullet e$.

(iv) The left component HX is precisely triangle (i) after application of the functor H. The right component A trivially commutes.

The remaining task is to show that (v) commutes. One way to do that is to show that all the $(He^\dagger + A) \cdot e$ together are jointly epic. Therefore, recall from Proposition 4.11 that HA is the colimit of the directed diagram D_{HA} of its finitely generated subobjects, in which both the connecting morphisms and the colimit injections are monic.

The rest of the proof is inspired by [AMV06, Proposition 2.18]. Consider such a f.g. subobject $z : Z \rightarrowtail HA$. As $\mathcal{B}(Z, -)$ preserves directed colimits of monos, z factors into some $s : Z \rightarrowtail HX$ and the colimit injection $\mathsf{in}_X : X \rightarrowtail A$ with X f.g. and

$$\begin{tikzcd}
Z \arrow[r, tail, "z"] \arrow[dr, tail, "s"'] & HA \\
& HX \arrow[u, tail, "H\mathsf{in}_X"']
\end{tikzcd}$$

commutes. Now define the equation morphism e as

$$e \equiv \; Z + X \xrightarrow{s + \mathsf{in}_X} HX + A \xrightarrow{H\mathsf{inr} + A} H(Z + X) + A.$$

Note that $e \cdot \mathsf{inr}$ always ends up in the A case and $e \cdot \mathsf{inl}$ always is in the $H(Z+X)$ case. Consider the diagram:

$$\begin{tikzcd}
Z+X \arrow[rr, "e^\dagger"] \arrow[d, "s+\mathsf{in}_X"'] \arrow[dd, bend right=60, "e"'] & & A \\
HX+A \arrow[d, "H\mathsf{inr}+A"'] & & \\
H(Z+X)+A \arrow[rr, "He^\dagger+A"'] & & HA+A \arrow[uu, "{[\alpha, A]}"']
\end{tikzcd}$$

So $e^\dagger \cdot \mathsf{inr} = (s + \mathsf{in}_X) \cdot \mathsf{inr} = \mathsf{in}_X$. For the left-hand component we have:

$$\begin{aligned}
e^\dagger \cdot \mathsf{inl} &= [\alpha, A] \cdot (He^\dagger + A) \cdot (H\mathsf{inr} + A) \cdot (s + \mathsf{in}_X) \cdot \mathsf{inl} \\
&= [\alpha, A] \cdot (He^\dagger + A) \cdot (H\mathsf{inr} + A) \cdot \mathsf{inl} \cdot s \\
&= [\alpha, A] \cdot \mathsf{inl} \cdot He^\dagger \cdot H\mathsf{inr} \cdot s \\
&= \alpha \cdot H(e^\dagger \cdot \mathsf{inr}) \cdot s = \alpha \cdot H\mathsf{in}_X \cdot s
\end{aligned}$$

In total $e^\dagger = [\alpha \cdot H\mathsf{in}_X \cdot s, \mathsf{in}_X]$ and for the diagonal:

$$\begin{aligned}
(He^\dagger + A) \cdot e &= (He^\dagger + A) \cdot (H\mathsf{inr} + A) \cdot (s + \mathsf{in}_X) \\
&= He^\dagger \cdot H\mathsf{inr} \cdot s + \mathsf{in}_X \\
&= H\mathsf{in}_X \cdot s + \mathsf{in}_X = z + \mathsf{in}_X
\end{aligned}$$

As the $(z)_{z:Z \rightarrowtail HA}$ are jointly epic and for each z all the compatible $(\mathsf{in}_X)_{\mathsf{in}_X : X \to A}$ are jointly epic, their sum $z + \mathsf{in}_X : Z + X \to HA + A$ is jointly epic as well. $\square$

So the fg-iterative algebras form a full subcategory of the category of all H-algebras. We denote the category of fg-iterative algebras by $\mathsf{Alg}_{\mathrm{it}} H$. The interesting question is the one about the initial object $\mathsf{Alg}_{\mathrm{it}} H$. The final coalgebra, which is in $\mathsf{Alg}_{\mathrm{it}} H$, is not the initial fg-iterative algebra in general. The goal of the remainder of the section is to show that the initial fg-iterative algebra is the locally finite fixpoint. Therefore, let $(\vartheta H, \ell)$ denote the final lfg coalgebra.

Theorem 4.58. *Every equation morphism $e : X \to HX + \vartheta H$ induces an lfg coalgebra*

$$\bar{e} \equiv \big(X + \vartheta H \xrightarrow{[e, \mathsf{inr}]} HX + \vartheta H \xrightarrow{HX + \ell} HX + H\vartheta H \xrightarrow{\mathsf{can}} H(X + \vartheta H)\big).$$

The only task is to show that $\bar{e}$ is lfg. So essentially for each $f : S \to X + \vartheta H$ where f is f.g. we have to find a coalgebra through which f factors, as required by Definition 4.23. Roughly this is done in two steps: firstly we construct the f.g. image of e in ϑH, secondly the f.g. image of f in ϑH, for the union P of these images, we construct a coalgebra structure on $X + P$ through which f factors. In order to get this kind of image factorization of f and e from the property of X being finitely generated, ϑH has to be expressed as a directed colimit of monos. This is done with the following lemmas before going into the detail of the proof of the theorem.

Lemma 4.59. *Let $\mathsf{Coalg}'_{\mathrm{fg}}$ be the full subdiagram of $\mathsf{Coalg}_{\mathrm{fg}}$ consisting of those coalgebras (A, a) where $a^\dagger : A \to \vartheta H$ is a monomorphism. Then the forgetful functor $U' : \mathsf{Coalg}'_{\mathrm{fg}} \to \mathcal{B}$ is a directed diagram of monos and filtered.*

Proof. At first, let us show that

$$\text{for all } (A, a) \text{ in } \mathsf{Coalg}_{\mathrm{fg}} \text{ there exists } (A', a') \text{ in } \mathsf{Coalg}'_{\mathrm{fg}} \text{ with } h : (A, a) \to (A', a'). \qquad (4.3)$$

This follows directly from the (strong epi,mono) factorization which lifts from $\mathcal{B}$ to $\mathsf{Coalg}_{\mathrm{fg}}$, see Proposition 4.20. So $a^\dagger : A \to \vartheta H$ factors into $h : A \twoheadrightarrow A'$ and $a'^\dagger : A' \rightarrowtail \vartheta H$. The strong epi h induces the structure $a' : A' \to HA'$ and proves that both h and $a'^\dagger$ are coalgebra homomorphisms. For the existence of upper bounds, which is required by the directedness, observe that coproducts exists in $\mathsf{Coalg}_{\mathrm{fg}}$, inducing upper bounds in $\mathsf{Coalg}'_{\mathrm{fg}}$ by (4.3).

For any homomorphisms $g, h : (A_1, a_1) \to (A_2, a_2)$ we have $a_2^\dagger \cdot g = a_1^\dagger = a_2^\dagger \cdot h$. As $a_2^\dagger$ is monic, $g = h$, i.e. there is at most one arrow in each hom set of $\mathsf{Coalg}'_{\mathrm{fg}}$, which means that U' is essentially small, a poset, and thus directed. As $a_1^\dagger$ is a mono, h is a mono as well, so U' is a directed diagram of monos. ☐

Lemma 4.60. *ϑH is the colimit of $U' : \mathsf{Coalg}'_{\mathrm{fg}} \to \mathcal{B}$.*

Proof. As (4.3) proves, the inclusion functor $V : \mathsf{Coalg}'_{\mathrm{fg}} \to \mathsf{Coalg}_{\mathrm{fg}}$ is a cofinal subdiagram. ϑH is the colimit of the forgetful functor $U : \mathsf{Coalg}_{\mathrm{fg}} \to \mathcal{B}$, so $\operatorname{colim} U = \operatorname{colim} UV = \operatorname{colim} U'$. ☐

Proof of Theorem 4.58. Consider the equation morphism $e : X \to HX + \vartheta H$. The functor $HX + (-)$ is finitary, so $HX + \vartheta H = \operatorname{colim}(Z \mapsto HX + UZ)$. By the previous lemma, $HX + \vartheta H$ is a directed diagram of monos. Hence the fact that X is finitely generated gives us a factorization through a $(V, v : V \to HV)$ in $\mathsf{Coalg}'_{\mathrm{fg}}$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e_0"'] & HX + \vartheta H \\
& HX + \underbrace{U(V,v)}_{=V} \arrow[u, "HX + v^\dagger"']
\end{tikzcd} \tag{4.4}$$

To prove the actual lfg property, assume some $f : S \to X + \vartheta H$ in ϑH, with S finitely generated. Analogously to e, f factors through some $(W, w : W \to HW)$ in $\mathsf{Coalg}'_{\mathrm{fg}}$:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "f_0"'] & X + \vartheta H \\
& X + U(W,w) \arrow[u, "X + w^\dagger"']
\end{tikzcd}$$

Define $(P, p) := (V, v) + (W, w)$ in $\mathsf{Coalg}_{\mathrm{fg}}$ – we do not need that $p^\dagger$ is monic, so we can stay in $\mathsf{Coalg}_{\mathrm{fg}}$. Let us define a coalgebra structure on $X + P$ and see that $X + p^\dagger$ is a coalgebra homomorphism:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "f_0"'] & X + \vartheta H \arrow[rr, "{[e, \mathsf{inr}]}"] \arrow[drr, phantom, "\text{(i)}"] & & HX + \vartheta H \arrow[r, "HX + \ell"] \arrow[dr, phantom, "\text{(ii)}"] & HX + H\vartheta H \arrow[r, "\mathsf{can}"] \arrow[dr, phantom, "\text{(iii)}"] & H(X + \vartheta H) \\
X + W \arrow[ur, "X + w^\dagger"] \arrow[r, "X + \mathsf{inr}"] & X + P \arrow[u, "X + p^\dagger"'] \arrow[r, "e_0 + P"] & (HX + V) + P \arrow[r, "{[HX + \mathsf{inl}, \mathsf{inr}]}"] & HX + P \arrow[u, "HX + p^\dagger"'] \arrow[r, "HX + p"] & HX + HP \arrow[u, "HX + Hp^\dagger"'] \arrow[r, "\mathsf{can}"] & H(X + P) \arrow[u, "H(X + p^\dagger)"']
\end{tikzcd}$$

Let us check the commutativity of the bottom triangle and the three squares.

- The triangle commutes because of the finality of ϑH.
- For the right-hand component P, (i) commutes trivially. For the left-hand component, recall that $e = (HX + v^\dagger) \cdot e_0$ by (4.4). By $v^\dagger = p^\dagger \cdot \mathsf{inl}$ we get the desired $e = (HX + p^\dagger \cdot \mathsf{inl}) \cdot e_0$.

- In the left component of (ii) are identities only. The right component of (ii) commutes because $p^\dagger : (P, p) \to (\vartheta H, \ell)$ is a coalgebra homomorphism.
- Recall that $\mathsf{can} = [H\mathsf{inl}, H\mathsf{inr}]$, so the left component reduces to $H\mathsf{inl} = H\mathsf{inl}$. For the right component, we have to verify $H\mathsf{inr} \cdot Hp^\dagger = H(X + p^\dagger) \cdot H\mathsf{inr}$. This holds because $H\mathsf{inr} : H \to H(X + -)$ is a natural transformation.

So $X + p^\dagger : X + P \to X + \vartheta H$ is indeed a coalgebra homomorphism. $X + P$ is finitely generated, hence $X + \vartheta H$ is an lfg coalgebra. ☐

Theorem 4.61. *The inverse of the structure of the locally finite fixpoint, namely the algebra $(\vartheta H, \ell^{-1})$, is fg-iterative.*

Proof. We are able to adapt the proof that final coalgebras are fg-iterative [Mil05, Example 2.5 (iii)] as follows. Assume an equation morphism $e : X \to HX + \vartheta H$ for ϑH. Define the equation morphism $\bar{e}$ just as in Theorem 4.58:

$$\bar{e} \equiv \big(X + \vartheta H \xrightarrow{[e,\mathsf{inr}]} HX + \vartheta H \xrightarrow{HX+\ell} HX + H\vartheta H \xrightarrow{\mathsf{can}} H(X + \vartheta H)\big).$$

Consider the unique morphism $\bar{e}^\dagger = [l, r] : (X + \vartheta H, \bar{e}) \to (\vartheta H, \ell)$ into the final lfg coalgebra. As the right-hand component of $\bar{e}$ essentially is ℓ, r must be the identity on ϑH. Now consider the following diagram for an *arbitrary* morphism $s : X \to \vartheta H$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "s"] \arrow[dr, phantom, "\text{(i)}"] & HX + \vartheta H \arrow[r, "HX + \ell"] \arrow[d, "Hs + \vartheta H"] & HX + H\vartheta H \arrow[r, "\mathsf{can}"] \arrow[dr, "{[Hs, H\vartheta H]}"] \arrow[d, phantom, "\text{(ii)}"] & H(X + \vartheta H) \arrow[d, "{H[s, \vartheta H]}"] \\
\vartheta H \arrow[rrr, bend right, "\ell"', "\cong"] & H\vartheta H + \vartheta H \arrow[l, "{[\ell^{-1}, \vartheta H]}"] \arrow[rr, "{[H\vartheta H, \ell]}"'] & & H\vartheta H
\end{tikzcd} \tag{4.5}$$

Note that the top right triangle of can and Hs always commutes, as well as the bottom triangle. The square (ii) commutes:

- For the left-hand component HX, the square reduces to the equality $Hs \cdot HX = H\vartheta H \cdot Hs$.
- For the right-hand component ϑH, it reduces to $H\vartheta H \cdot \ell = \ell \cdot \vartheta H$.

So all parts of (4.5) except (i) commute. Now consider the following list of equivalences:

s is a solution of e in ϑH.

$\Leftrightarrow$ The square (i) commutes.

$\Leftrightarrow$ The entire diagram (4.5) commutes.

$\Leftrightarrow$ $[s, r] : (X + \vartheta H, \bar{e}) \to (\vartheta H, \ell)$ is a coalgebra homomorphism.

$\Leftrightarrow$ $[s, r] = [l, r] = \bar{e}^\dagger$.

Reading this from bottom to top gives us the existence of a solution $s = l$. Reading this from top to bottom for another solution $\bar{s}$, gives us that $\bar{s} = s$ by the uniqueness of $\bar{e}^\dagger$, hence $(\vartheta H, \ell^{-1})$ is fg-iterative. ☐

We have seen that the coalgebra, which is final for $\mathsf{Coalg}_{\mathrm{fg}}$, can be considered as an fg-iterative algebra. The converse direction is also true, namely, every fg-iterative algebra is "final" for all coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}$ in the following sense.

Theorem 4.62. *For an fg-iterative algebra $(A, \alpha : HA \to A)$ and a coalgebra $e : X \to HX$ from $\mathsf{Coalg}_{\mathrm{fg}}$ there is a unique $\mathcal{B}$-morphism $u_e : X \to A$ such that $u_e = \alpha \cdot Hu_e \cdot e$.*

$$\begin{tikzcd}
X \arrow[r, dashed, "\exists! u_e"] \arrow[d, "e"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\
HX \arrow[r, "Hu_e"] & HA \arrow[u, "\alpha"']
\end{tikzcd}$$

Proof. Consider the equation morphism $\mathsf{inl} \cdot e : X \to HX + A$. For an arbitrary morphism $s : X \to A$, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "s"] \arrow[d, "e"] & & A \\
HX \arrow[r, "\mathsf{inl}"] \arrow[rrd, bend right, "Hs"'] & HX + A \arrow[r, "Hs + A"] \arrow[d, phantom, "\circlearrowleft"] & HA + A \arrow[u, "{[\alpha, A]}"'] \\
& & HA \arrow[u, "\mathsf{inl}"'] \arrow[uu, bend right=60, "\alpha"']
\end{tikzcd}$$

The lower part and the right-hand part always commute. But for the commutativity of the whole diagram consider the following sequence of equivalences:

s is a solution of $\mathsf{inl} \cdot e$ in A.

$\Leftrightarrow$ The upper square commutes.

$\Leftrightarrow$ $s = [\alpha, A] \cdot \mathsf{inl} \cdot Hs \cdot e$

$\Leftrightarrow$ $s = \alpha \cdot Hs \cdot e$

So by the existence and the uniqueness of a solution of $\mathsf{inl} \cdot e$ in the fg-iterative algebra A, we get the desired morphism $u_e : X \to A$ with $u_e = \alpha \cdot Hu_e \cdot e$ and its uniqueness, by reading the equivalences from top or from bottom respectively. $\square$

Note that the theorem can be read with the two universal quantifications swapped: each $e : X \to HX$ from $\mathsf{Coalg}_{\mathrm{fg}}$ is initial for the fg-iterative algebras in the sense that for each (A, α) there is a unique $u_e : X \to A$ such that $u_e = \alpha \cdot Hu_e \cdot e$. We can generalize Theorem 4.62 to lfg coalgebras the following way:

Theorem 4.63. *For an fg-iterative algebra $(A, \alpha : HA \to A)$ and an lfg coalgebra $e : X \to HX$ there is a unique $\mathcal{B}$-morphism $u_e : X \to A$ such that $u_e = \alpha \cdot Hu_e \cdot e$.*

Proof. By Proposition 4.34, $e : X \to HX$ is the union of the diagram D of its subcoalgebras $s : S \to HS$ with S finitely generated. Denote the corresponding colimit injections by $\mathsf{in}_s : (S, s) \to (X, e)$. Each such s induces a unique morphism $u_s : S \to A$ with

$$u_s = \alpha \cdot Hu_s \cdot s. \tag{4.6}$$

For any coalgebra homomorphism $h : (R, r) \to (S, s)$ in $\mathsf{Coalg}_{\mathrm{fg}}$ the diagram

$$\begin{tikzcd}
R \arrow[r, "h"] \arrow[d, "r"'] & S \arrow[r, "u_s"] \arrow[d, "s"'] & A \\
HR \arrow[r, "Hh"'] & HS \arrow[r, "Hu_s"'] & HA \arrow[u, "\alpha"']
\end{tikzcd}$$

commutes, because h is a coalgebra homomorphism and because of the property of u_s. So $u_r = u_s \cdot h$. In other words, A together with the morphisms $(u_s : S \to A)_{s:S\to HS \text{ lfg}}$ form a cocone for D in $\mathcal{B}$. This induces a unique morphism $u_e : X \to A$.

For each $s : S \to HS$, $\mathsf{in}_s : S \to X$ is a coalgebra homomorphism. Furthermore, we have is $u_s = u_e \cdot \mathsf{in}_s$ in $\mathcal{B}$ by the universal property of X. So every part except possibly (ii) of the diagram

$$
\begin{tikzcd}
S \arrow[r, "\mathsf{in}_s"] \arrow[d, "s"'] \arrow[rr, bend left, "u_s"] \arrow[dr, phantom, "(\mathrm{i})\ \circlearrowleft"] & X \arrow[r, "u_e"] \arrow[d, "e"'] \arrow[u, phantom, "\circlearrowleft"] \arrow[dr, phantom, "(\mathrm{ii})"] & A \\
HS \arrow[r, "H\mathsf{in}_s"'] \arrow[rr, bend right, "Hu_s"'] & HX \arrow[r, "Hu_e"'] \arrow[d, phantom, "\circlearrowleft"] & HA \arrow[u, "\alpha"']
\end{tikzcd}
$$

commutes, as indicated. In particular the outer square square commutes which gives

$$\alpha \cdot Hu_e \cdot e \cdot \mathsf{in}_s = u_e \cdot \mathsf{in}_s \text{ for every f.g. subcoalgebra } (S, s) \text{ of } (X, e).$$

As the colimit injections in_s are jointly epic, (ii) commutes.

Conversely every $\mathcal{B}$-morphism $\tilde{u}_e : X \to A$ making (ii) commute, makes the bigger square (i)+(ii) commute and defines a family of morphisms $\tilde{u}_e \cdot \mathsf{in}_s : S \to A$ having the property (4.6) each. So by the uniqueness of the $u_s : S \to A$, we get $u_s = \tilde{u}_e \cdot \mathsf{in}_s$. Using again that the in_s are jointly epic, reduces the equation

$$u_e \cdot \mathsf{in}_s = u_s = \tilde{u}_e \cdot \mathsf{in}_s$$

to the desired uniqueness of u_e, namely $u_e = \tilde{u}_e$. □

Applying Theorem 4.63 to the final lfg coalgebra $(\vartheta H, \ell)$ we get as a corollary:

Corollary 4.64. *The locally finite fixpoint $(\vartheta H, \ell : \vartheta H \to H\vartheta H)$ is the initial fg-iterative algebra, i.e. for each fg-iterative algebra $(A, \alpha : HA \to A)$ there is a unique $h : \vartheta H \to A$ such that the following square commutes:*

$$
\begin{tikzcd}
\vartheta H \arrow[r, dashed, "\exists! h"] \arrow[d, shift right, "\ell"'] & A \\
H\vartheta H \arrow[r, "h"'] \arrow[u, shift right, "\ell^{-1}"'] & HA \arrow[u, "\alpha"']
\end{tikzcd}
$$

In the big picture we have that each lfg coalgebra is initial for all fg-iterative algebras and each fg-iterative algebra is final for the lfg coalgebras in the above sense.

If we consider any isomorphism $HI \xrightarrow{i} I$ where (I, i) is an fg-iterative algebra and (I, i^{-1}) is an lfg coalgebra, then (I, i) is isomorphic to $(\vartheta H, \ell)$. So any such (I, i) is both the final lfg coalgebra and the initial fg-iterative algebra. This means that the unique morphisms from Theorem 4.63 factor through this fixpoint.

So given an lfg coalgebra (C, c) and an fg-iterative algebra (A, α), we have (i) the induced morphism u_C as in Theorem 4.63, (ii) the unique coalgebra homomorphism $g : C \to \vartheta H$, (iii) the unique algebra homomorphism $h : \vartheta H \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, dashed, "g"] \arrow[d, "c"'] \arrow[rr, dashed, bend left, "u_C"] & \vartheta H \arrow[r, dashed, "h"] \arrow[d, shift right, "\ell"'] & A \\
HC \arrow[r, "Hg"] \arrow[rr, bend right, "Hu_C"'] & H\vartheta H \arrow[r, "Hh"] \arrow[u, shift right, "\ell^{-1}"'] & HA \arrow[u, "\alpha"']
\end{tikzcd}$$

In other words: the *locally finite fixpoint* is the unique connecting piece between lfg coalgebras and fg-iterative algebras.

5 Applications

The locally finite fixpoint applies to all scenarios in which behaviours of coalgebras with a finitely generated carrier are collected. Of course, this includes all applications in which the rational fixpoint is considered. More interesting are examples, in which finitely generated and finitely presentable do not coincide. In some of those, the image of the rational fixpoint in the final coalgebra has been considered up to now. Alternatively, authors have spoken about the behaviours of the coalgebras with f.g. carrier in a informal way. Both cases now can be uniformly treated using the formal framework given by the LFF.

Let us first recap those scenarios, before we see how the LFF helps us in each of these.

5.1 About Algebras

A scenario gets interesting, if f.g. and f.p. do not coincide. This is often the case in algebraic categories like Monoids, Groups, or the category of finitary monads. An algebraic category is a category of Eilenberg-Moore algebras for a *monad*:

Definition 5.1. For a category $\mathcal{B}$, a *monad* is an endofunctor $T : \mathcal{B} \to \mathcal{B}$ together with two natural transformations $\eta : \mathrm{Id}_{\mathcal{B}} \to T$, the *unit*, and $\mu : T^2 \to T$, the *monad multiplication*, such that the following diagrams commute:

$$\begin{tikzcd}
T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "T"] \\
& T &
\end{tikzcd}$$

Monads stand in close correspondence to *adjoint functors*: Every pair of adjoint functors gives rise to a monad and every monad can be factorized into a pair of adjoint functors, in possibly multiple ways. Furthermore, the unit of the adjunction becomes the unit of the monad and vice versa. For more details see [Awo10, Chapter 10]

Definition 5.2. For a monad T on $\mathcal{B}$, we denote by $\mathcal{B}^T$ the Eilenberg-Moore category, consisting of T-algebras (A, α), with $\alpha : TA \to A$ a morphism in $\mathcal{B}$ such that

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "A"'] & TA \arrow[d, "\alpha"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\
TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\tag{5.1}$$

commute. An algebra homomorphism $h : (A, \alpha) \to (B, \beta)$ is a morphism $h : A \to B$ of $\mathcal{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
TA \arrow[r, "\alpha"] \arrow[d, "Th"'] & A \arrow[d, "h"] \\
TB \arrow[r, "\beta"'] & B
\end{tikzcd}$$

A monad $T : \mathcal{B} \to \mathcal{B}$, can be factorized through $\mathcal{B}^T$ into two adjoint functors $F \dashv U$, with

$$FX = (TX, \mu_X : TTX \to TX) \qquad\qquad U(A, \alpha) = A. \tag{5.2}$$

Example 5.3. Many prominent categories are equivalent to $\mathcal{B}^T$ for suitable T and $\mathcal{B}$, where $\mathcal{B}$ mostly is Set:

- The category of monoids Mon is obtained by taking Set^T, where $TX = X^*$ maps a set X to the set of finite words over X. Here, the unit $\eta_X : X \to TX$ maps elements $x \in X$ to the single letter word $x \in X^*$ and the multiplication $\mu_X : X^{**} \to X^*$ concatenates a word of words over X to a word over X.
- The category of groups is obtained similarly as Set^T, where T is the construction of the free group.
- For a category $\mathcal{B}$ with coproducts and an object B, one has a free construction of a B-pointed object, i.e. an adjuction

  $$F : \mathcal{B} \rightleftarrows (B/\mathcal{B}) : U$$

  with $F : X \mapsto (X{+}B, \mathsf{inr})$ and $U : (X, b : B \to X) \mapsto X$ and with the unit $\mathsf{inl} : X \to UFX$. The one-to-one correspondence of the adjunction is just the universal property of the coproduct in disguise:

  $$\begin{array}{c} (X+B, \mathsf{inr}) \overset{\exists! \bar{f}}{\dashrightarrow} (Y, y : B \to Y) \\[1em] \begin{tikzcd} X+B \arrow[r, "U\bar{f}"] & Y \\ X \arrow[u, "\mathsf{inl}"] \arrow[ur, "\forall f"'] \arrow[u, phantom, shift left=3ex, "\circlearrowleft"] \end{tikzcd} \end{array} \quad \Longleftrightarrow \quad \begin{tikzcd} B \arrow[d, "\mathsf{inr}"'] \arrow[dr, "\forall y"] & \\ X+B \arrow[r, dashed, "\exists!"] \arrow[ur, phantom, shift right=1ex, "\circlearrowleft"] & Y \\ X \arrow[u, "\mathsf{inl}"] \arrow[ur, "\forall f"'] \arrow[u, phantom, shift left=3ex, "\circlearrowleft"] & \end{tikzcd}$$

  To be even more explicit, the multiplication of the corresponding monad T is just

  $$\mu_X \equiv \big(TTX = X + B + B \xrightarrow{X+[B,B]} X + B\big).$$

For another example with a base category different than Set, see Definition 5.54 later.

Remark 5.4. In this work, we will focus on finitary monads, i.e. monads whose underlying functor is finitary. Finitary monads on Set correspond to algebraic theories generated by operations of finite arity and arbitrarily many equations, see [Man76, Theorem 4.25] or [ARV10, Appendix A, especially Theorem A.21].

Remark 5.5. In general, right adjoints preserve limits, moreover it is not difficult to show that limits in $\mathcal{B}^T$ are created by the forgetful U. But generally, only those colimits are created by U, which are preserved by T. In particular, filtered colimits in $\mathcal{B}^T$ are created by U, if T is finitary, i.e. filtered colimits in $\mathcal{B}^T$ are computed on the level of $\mathcal{B}$.

If our base category $\mathcal{B}$ is lfp and T is finitary, then $\mathcal{B}^T$ is lfp. The forgetful functor $U : \mathcal{B}^T \to \mathcal{B}$ has a left adjoint, namely mapping objects to the free T-algebras over it. This free construction maps f.p. objects from $\mathcal{B}$ to f.p. T-algebras in $\mathcal{B}^T$, see [AR94, Corollary 2.75].

In Set^T, epimorphisms are not necessarily surjective, e.g. the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic in the category of Rings and Semigroups, see [AHS04, Example 7.40(5)]. But extremal and, in particular, strong epimorphisms are surjective T-algebra homomorphisms: like all Set-functors, T preserves epimorphisms, so the (epi,mono)-factorization of a T-algebra homomorphism lifts to Set^T. And if the T-algebra homomorphism was an extremal epi in Set^T, the monic component of the factorization is an iso.

Example 5.6. For $TX = X^*$, the free monoid monad, coproduct injections are not jointly surjective, i.e. for $\mathsf{inl} : TX \to TX + TY \leftarrow TY : \mathsf{inr}$, X and Y non-empty, the Set-functions $U\mathsf{inl}, U\mathsf{inr}$ are not jointly epic. With $x \in X$, $y \in Y$, the element $x \cdot y \in TX + TY = T(X + Y)$ is neither in the image of inl nor in that of inr.

Proposition 5.7. *Let $(\mathsf{in}_i : D_i \to C)_{i \in I}$ be the colimit injections of a filtered colimit in Set^T and T finitary. Then the $U\mathsf{in}_i$ are jointly epic in Set.*

Proof. As T is finitary, filtered colimits in Set^T are created by the forgetful $U : \mathsf{Set}^T \to \mathsf{Set}$. So the $U\mathsf{in}_i : UD_i \to UC$ are a colimit cocone and hence jointly epic. $\square$

Interestingly, the monads itself form a category. But before we are able to give the correct notion of homomorphism, the following auxiliary definition is required:

Definition 5.8 (Godement product)**.** For natural transformations $\varphi : E \to F$ and $\sigma : G \to H$, the *Godement product* of φ and σ is a natural transformation

$$\varphi * \sigma : EG \to FH$$

defined by

$$\varphi * \sigma \equiv \left(EG \xrightarrow{E\sigma} EH \xrightarrow{\varphi H} FH \right) \equiv \left(EG \xrightarrow{\varphi G} FG \xrightarrow{F\sigma} FH \right).$$

Note that the two compositions are point-wise the same $(\varphi * \sigma)_X$, because of the naturality of φ for the morphism $\sigma_X : GX \to HX$:

$$\begin{tikzcd}
EGX \arrow[r, "\varphi_{GX}"] \arrow[d, "E\sigma_X"'] \arrow[dr, phantom, "\circlearrowleft"] & FGX \arrow[d, "F\sigma_X"] \\
EHX \arrow[r, "\varphi_{HX}"'] & FHX
\end{tikzcd}$$

Definition 5.9. A *monad morphism* from (S, η^T, μ^T) to (T, η^T, μ^T) is a natural transformation $\varphi : S \to T$ such that

$$\begin{tikzcd}
\mathrm{Id} \arrow[r, "\eta^S"] \arrow[dr, "\eta^T"'] & S \arrow[d, "\varphi"] & S^2 \arrow[l, "\mu^S"'] \arrow[d, "\varphi * \varphi"] \\
& T & T^2 \arrow[l, "\mu^T"]
\end{tikzcd} \tag{5.3}$$

commutes.

Definition 5.10. By $\mathsf{Fun_f}(\mathcal{B})$, denote the category of finitary endofunctors on $\mathcal{B}$ together with natural transformations.

Definition 5.11. The finitary monads on $\mathcal{B}$ and monad morphisms form the category $\mathsf{Mnd_f}(\mathcal{B})$.

Remark 5.12. $\mathsf{Fun_f}(\mathcal{B})$ is lfp. Colimits are built object-wise in $\mathcal{B}$. $\mathsf{Mnd_f}(\mathcal{B})$ is lfp too, In $\mathsf{Mnd_f}(\mathcal{B})$, the monomorphisms are precisely the monad morphisms with monic components. Furthermore, all monos in $\mathsf{Mnd_f}(\mathcal{B})$ are monic in $\mathsf{Fun_f}(\mathcal{B})$.

5.2 Functor Liftings

In the applications of the LFF we will consider functors on Eilenberg-Moore categories. One way to define a functor between Eilenberg-Moore categories $\mathcal{C}^T \to \mathcal{D}^M$ is by defining its behaviour on the underlying objects and morphisms of $\mathcal{C}$ and then lift this to algebras:

Definition 5.13 (Lifting)**.** For a functor $H : \mathcal{C} \to \mathcal{D}$ and two monads $T : \mathcal{C} \to \mathcal{C}$ and $M : \mathcal{D} \to \mathcal{D}$ a *lifting* of H is a functor $\bar{H} : \mathcal{C}^T \to \mathcal{D}^M$ such that

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "\bar{H}"] \arrow[d, "U^T"'] & \mathcal{D}^M \arrow[d, "U^M"] \\ \mathcal{C} \arrow[r, "H"] & \mathcal{D} \end{tikzcd}$$

where $U^T : \mathcal{C}^T \to \mathcal{C}$ and $U^M : \mathcal{D}^M \to \mathcal{D}$ denote the obvious forgetful functors.

Definition 5.14. For a functor $H : \mathcal{C} \to \mathcal{D}$ and two monads $T : \mathcal{C} \to \mathcal{C}$ and $M : \mathcal{D} \to \mathcal{D}$ a natural transformation $\lambda : MH \to HT$ is called a *distributive law*, if the following diagrams commute:

$$\begin{tikzcd} H \arrow[d, "\eta^M H"'] \arrow[dr, "H\eta^T"] & \\ MH \arrow[r, "\lambda"] & HT \end{tikzcd} \qquad \begin{tikzcd} MMH \arrow[r, "M\lambda"] \arrow[d, "\mu^M H"'] & MHT \arrow[r, "\lambda T"] & HTT \arrow[d, "H\mu^T"] \\ MH \arrow[rr, "\lambda"] & & HT \end{tikzcd}$$

It is well-known that these two definitions are in bijective correspondence:

Theorem 5.15 ([Joh75, Lemma 1] and [App65])**.** *For a functor $H : \mathcal{C} \to \mathcal{D}$ and monads $T : \mathcal{C} \to \mathcal{C}$ and $M : \mathcal{D} \to \mathcal{D}$, liftings $\bar{H} : \mathcal{C}^T \to \mathcal{D}^M$ are in one-to-one correspondence to distributive laws $\lambda : MH \to HT$.*

It is also well-known, that for given a functor $H : \mathcal{C} \to \mathcal{D}$ the lifting $\bar{H} : \mathcal{C}^T \to \mathcal{D}^M$ is not unique. We will see later in Example 5.19, how different choices of liftings affect our application.

But though the lifting is not unique, it is already uniquely determined by its restriction to free algebras.

Definition 5.16 (Kleisli Category)**.** Denote by $\mathcal{C}_T$ the full subcategory of $\mathcal{C}^T$, containing only the free algebras $(TX, \mu_X : TTX \to TX)_{X \in \mathcal{C}}$.

Another commonly used and equivalent definition of the Kleisli category can be found in e.g. [Awo10, Exercise 10.6.9].

Lemma 5.17. *Assume for a functor $H : \mathcal{C} \to \mathcal{D}$ and monads $T : \mathcal{C} \to \mathcal{C}$, $M : \mathcal{D} \to \mathcal{D}$, and two liftings $\bar{H}, \widetilde{H} : \mathcal{C}^T \to \mathcal{D}^M$ of H, that the liftings behave identically on free algebras; then $\bar{H} = \widetilde{H}$.*

Proof. Given a T-algebra $(A, \alpha : TA \to A)$ and its images $(HA, \bar{\alpha} : MHA \to HA)$ and $(HA, \widetilde{\alpha} : MHA \to HA)$ under $\bar{H}$ and $\widetilde{H}$ respectively. As both functors are liftings of H we know that both algebras $\bar{\alpha}$ and $\widetilde{\alpha}$ are carried by HA and that $\bar{H}h = \widetilde{H}h = Hh$ for any morphism h in $\mathcal{C}^T$. The algebra structure $\alpha : TA \to A$ is a homomorphism $(TA, \mu_A) \to (A, \alpha)$ by (5.1).

$$\begin{tikzcd}
 & & MHA \arrow[dll, "MHA"'] \arrow[d, "MH\eta^T_A"] \arrow[drr, "MHA"] & & \\
MHA \arrow[ddd, "\bar{\alpha}"'] & & MHTA \arrow[ll, "MH\alpha"] \arrow[rr, "MH\alpha"'] \arrow[dd, bend right, "\bar{\mu}^T_A"'] \arrow[dd, bend left, "\widetilde{\mu}^T_A"] \arrow[dd, phantom, "="] & & MHA \arrow[ddd, "\widetilde{\alpha}"] \\
 & & & & \\
 & & HTA \arrow[dll, "H\alpha"'] \arrow[drr, "H\alpha"] & & \\
HA \arrow[rrrr, equal] & & & & HA
\end{tikzcd}$$

The upper triangles commute because of the unit law for α, the left-hand square because $\bar{H}\alpha : (HTA, \bar{\mu}^T_A) \to (HA, \bar{\alpha})$ is a homomorphism, and the right-hand square because $\widetilde{H}\alpha : (HTA, \widetilde{\mu}^T_A) \to (HA, \widetilde{\alpha})$ is one as well. By assumption we have $\bar{\mu}^T_A = \widetilde{\mu}^T_A$, and hence $\bar{\alpha} = \widetilde{\alpha}$. $\square$

Moreover, the behaviour of a lifting is not only uniquely determined by its behaviour on free algebras: defining a lifting only on free algebras, i.e. on $\mathcal{C}_T$, induces a unique lifting to $\mathcal{C}^T$.

Theorem 5.18. *For a functor $H : \mathcal{C} \to \mathcal{D}$ and monads $T : \mathcal{C} \to \mathcal{C}$, $M : \mathcal{D} \to \mathcal{D}$, the following notions are in pairwise one-to-one corespondence:*

1. *Liftings $\bar{H} : \mathcal{C}^T \to \mathcal{D}^M$.*
2. *Distribuitive laws $\lambda : MH \to HT$.*
3. *Liftings $\hat{H} : \mathcal{C}_T \to \mathcal{D}^M$ of H, i.e. a functor $\hat{H}$ sending free algebras (TX, μ_X) to M-algebras (HTX, t_X) with*

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, "\hat{H}"] \arrow[d, "U_T"'] & \mathcal{D}^M \arrow[d, "U^M"] \\
\mathcal{C} \arrow[r, "H"] & \mathcal{D}
\end{tikzcd} \tag{5.4}$$

4. *M-algebra structures $t_X : MHTX \to HTX$ on every HTX, $X \in \mathcal{C}$, natural in X and for which the following commutes:*

$$\begin{tikzcd}
MHTTX \arrow[r, "MH\mu^T_X"] \arrow[d, "t_{TX}"'] & MHTX \arrow[d, "t_X"] \\
HTTX \arrow[r, "H\mu^T_X"] & HTX
\end{tikzcd} \tag{5.5}$$

More detailed, one gets $\hat{H}$ from $\bar{H}$ by restricting its domain to free algebras and from a given lifting $\hat{H} : \mathcal{C}_T \to \mathcal{D}^M$, we first get a distributive law

$$\lambda_X \equiv \big(MHX \xrightarrow{MH\eta^T_X} MHTX \xrightarrow{t_X} HTX\big), \tag{5.6}$$

and secondly a lifting $\bar{H}$, sending an algebra $TA \xrightarrow{\alpha} A$ to

$$h_\alpha \equiv \big(MHA \xrightarrow{MH\eta^T_A} MHTA \xrightarrow{t_A} HTA \xrightarrow{H\alpha} HA\big). \tag{5.7}$$

Proof. The correspondence 1$\Leftrightarrow$2 is Theorem 5.15.

For the implication 3$\Rightarrow$4 note that $\hat{H}$ sends T-algebra homomorphisms $h : TX \to TY$ to M-algebra homomorphisms $\hat{H}h : HTX \to HTY$. In particular for the homomorphism μ^T_X :

$TTX \to TX$, we get (5.5) and for $h = Tf$ for $f : X \to Y$ a morphism $\mathcal{C}$ we get that $t_X : MHTX \to HTX$ is natural in X.

For the converse 4⇒3, one defines $\hat{H}(TX, \mu_X) = (HTX, t_X)$. Then it only remains to show that any T-algebra homomorphism $h : (TX, \mu_X) \to (TY, \mu_Y)$ is indeed mapped to an M-algebra homomorphism $(HTX, t_X) \to (HTY, t_Y)$. The adjunction $\mathcal{C}^T \rightleftarrows \mathcal{C}$ induces a $\mathcal{C}$-morphism $h' : X \to TY$ with $U^T h \cdot \eta^T_X = h'$. By the two properties of condition 4, $H\mu^T_Y \cdot HTh' : (HTX, t_X) \to (HTY, t_Y)$ is a M-algebra homomorphism.

To prove the 3⇒1, consider $\hat{H} : \mathcal{C}_T \to \mathcal{D}^M$ and check that (5.7) indeed defines a functor $\bar{H} : \mathcal{C}^T \to \mathcal{D}^T$. The unit law of h_α holds because that of (i) the M-algebra t_A and (ii) the T-algebra α:

$$
\begin{tikzcd}
HA \arrow[r, "\eta^M_{HA}"] \arrow[d, "H\eta^T_A"'] \arrow[dr, phantom, "\text{naturality of } \eta^M \text{ for } H\eta^T_A"] \arrow[dddr, bend right, "HA"'] & MHA \arrow[d, "MH\eta^T_A"] \arrow[ddd, bend left=60, "h_\alpha"] \\
HTA \arrow[r, "\eta^M_{HTA}"] \arrow[dr, "HTA"'] \arrow[dr, phantom, shift left=3, "\text{(i)}"] \arrow[ddr, phantom, shift right=2, "\text{(ii)}"] & MHTA \arrow[d, "t_A"] \\
 & HTA \arrow[d, "H\alpha"] \\
 & HA
\end{tikzcd}
$$

As an intermediate result, observe that $\bar{H}$ sends the algebra $\mu^T_A : TTA \to TA$ to $h_{\mu^T_A} = t_A$, because μ^T_A is also a T-algebra homomorphism and thus $\hat{H}\mu^T_A$ an M-algebra homomorphism:

$$
\begin{tikzcd}
MHTA \arrow[r, "MH\eta^T_{TA}"] \arrow[rrr, bend left, "h_{\mu^T_A}"] \arrow[drr, bend right, "MHTA"'] \arrow[dr, phantom, "\circlearrowleft"] & MHTTA \arrow[r, "t_{TA}"] \arrow[dr, "MH\mu^T_A"'] & HTTA \arrow[r, "H\mu^T_A"] \arrow[d, phantom, "\substack{\text{homomorphism} \\ \hat{H}\mu^T_A}"] & HTA \\
 & & MHTA \arrow[ur, "t_A"'] &
\end{tikzcd}
\tag{5.8}
$$

For the multiplication law for h_α, exploit that (i) t_A is an M-algebra, (ii) $\hat{H}T\alpha : HTTA \to HTA$ is an M-algebra homomorphism, and (iii) α is a T-algebra:

$$
\begin{tikzcd}
MMHA \arrow[r, "MMH\eta^T_A"] \arrow[rrrr, bend left, "Mh_\alpha"] \arrow[dd, "\mu^M_{HA}"'] \arrow[ddr, phantom, "\substack{\text{naturality of} \\ \mu^M \text{ for } H\eta^T_A}"] & MMHTA \arrow[r, "Mt_A"] \arrow[dd, "\mu^M_{HTA}"] \arrow[ddr, phantom, "\text{(i)}"] & MHTA \arrow[rr, "MH\alpha"] \arrow[dr, "MH\eta^T_{TA}"] \arrow[ddd, "t_A"'] \arrow[ddr, phantom, "(5.8)"] & & MHA \arrow[d, "MH\eta^T_A"] \arrow[ddd, bend left=60, "h_\alpha"] \arrow[dl, phantom, "\substack{\text{naturality of} \\ MH\eta^T \text{ for } \alpha}"] \\
 & & & MHTTA \arrow[r, "MHT\alpha"'] \arrow[d, "t_{TA}"] \arrow[dr, phantom, "\text{(ii)}"] & MHTA \arrow[d, "t_A"] \\
MHA \arrow[r, "MH\eta^T_A"] \arrow[rrrr, bend right=40, "h_\alpha"'] & MHTA \arrow[dr, "t_A"] & & HTTA \arrow[r, "HT\alpha"] \arrow[dl, "H\mu^T_A"'] \arrow[dr, phantom, "\text{(iii)}"] & HTA \arrow[d, "H\alpha"] \\
 & & HTA \arrow[rr, "H\alpha"] & & HA
\end{tikzcd}
$$

So h_α is indeed an M-algebra structure on HA. For the functorality, consider a T-algebra homomorphism $h : (A, \alpha) \to (B, \beta)$. Then Hh is an M-algebra homomorphism:

$$\begin{tikzcd}
MHA \arrow[r, "MH\eta^T_A"] \arrow[rrr, bend left, "h_\alpha"] \arrow[d, "MHh"'] & MHTA \arrow[r, "t_A"] \arrow[d, phantom, "\substack{\text{naturality of}\\ t \cdot MH\eta^T : MH \to HT \text{ for } h}"] & HTA \arrow[r, "H\alpha"] \arrow[d, "HTh"'] \arrow[dr, phantom, "\substack{h \text{ homo-}\\ \text{morphism}}"] & HA \arrow[d, "Hh"] \\
MHB \arrow[r, "MH\eta^T_B"'] \arrow[rrr, bend right, "h_\beta"'] & MHTB \arrow[r, "t_B"'] & HTB \arrow[r, "H\beta"'] & HB
\end{tikzcd}$$

In total we have a lifting $\bar{H} : \mathcal{C}^T \to \mathcal{D}^T$ of H.

For $1 \Rightarrow 3$, just restrict $\bar{H}$ to free algebras. The two translations $3 \Rightarrow 1$ and $1 \Rightarrow 3$ are mutually inverse:

1. When constructing $\bar{H} : \mathcal{C}^T \to \mathcal{D}^T$ from a given $\hat{H} : \mathcal{C}_T \to \mathcal{D}^T$ and restricting $\bar{H}$ again to $\mathcal{C}_T$, one obtains the original $\hat{H}$, by (5.8).
2. When first restricting a given lifting $\bar{H} : \mathcal{C}^T \to \mathcal{D}^T$ to $\hat{H} : \mathcal{C}_T \to \mathcal{D}^T$ and extending this back to another $\widetilde{H} : \mathcal{C}^T \to \mathcal{D}^T$, we know that $\bar{H}$ and $\widetilde{H}$ behave identically on free algebras by (5.8) and thus by Lemma 5.17 $\bar{H} = \widetilde{H}$. □

5.3 Generalized Determinization

Given $H = 2 \times (-)^\Sigma$ on Set, we know that the final H-coalgebra consists of formal languages. In this section, we will consider coalgebras of the form

$$X \longrightarrow HTX = 2 \times TX^\Sigma,$$

where X is a finite set and T is a monad. So our HT-coalgebras consist of a set of states X. Each state $x \in X$ has an output from 2 (e.g. "accepting" or "rejecting"), and for every input symbol $\sigma \in \Sigma$ an successor state from X under some monadic side-effect w.r.t. T. Depending on the choice of T, the side-effect could consist of

- non-deterministically choosing one successor state from a set of possible successor states,
- pushing a value on a stack,
- moving the head along a Turing tape and writing/reading the cell under the head.

For an arbitrary T, our goal is to give the H-coalgebra carried by class of formal languages that can be described by such T-automata. The key here is the so called *generalized powerset construction* (or *generalized determinization*), which can be combined with the LFF in order to obtain the desired H-coalgebra.

5.3.1 What has been done already

In [SBBR13], Silva et al. show how to generalize the powerset construction of non-deterministic finite state machines to coalgebras in general.

Let us recall their basic idea first. As known previously, one can describe a non-deterministic finite-state machine with a set of states X, input symbols Σ, and output symbols B as a coalgebra

$$x : X \longrightarrow B \times (\mathcal{P}_{\mathsf{f}} X)^\Sigma$$

This can be generalized by rewriting the codomain of x as

$$x : X \longrightarrow HTX, \quad \text{with } TX = \mathcal{P}_{\mathsf{f}}X \text{ and } H = B \times (-)^{\Sigma}.$$

The freshly introduced functor H represents the basic notion of in- and output – the *transition type*. The functor, which is required to be a monad, T encodes the structure for successors, given already a concrete input symbol – the *computational type*.

They were able to generalize the powerset construction to this general setting, for a given functor H and a monad T. They furthermore required H to have a T-algebra lifting. In words, a T-algebra lifting of H is an endofunctor on the Eilenberg-Moore category $H^T : \mathsf{Set}^T \to \mathsf{Set}^T$ such that

$$\begin{tikzcd}
\mathsf{Set}^T \arrow[r, "H^T"] \arrow[d, "U"'] & \mathsf{Set}^T \arrow[d, "U"] \\
\mathsf{Set} \arrow[r, "H"] & \mathsf{Set}
\end{tikzcd}$$

commutes. With a lifting H^T, we have the correspondence of $F \dashv U$, $T = U \cdot F$:

$$\frac{\dfrac{\dfrac{X \longrightarrow HTX}{X \longrightarrow HUFX}}{X \longrightarrow UH^TFX}}{FX \longrightarrow H^TFX}$$

Applying U gives us a map $x^\sharp : TX \to HTX$ for any $x : X \to HTX$ such that $x = x^\sharp \cdot \eta_X$, where η is the unit of the monad T.

Having a map $x^\sharp : TX \to H(TX)$ induces a unique coalgebra homomorphism into the final F-coalgebra $\tau : \nu F \to F\nu F$:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "x"'] & TX \arrow[r, dashed, "x^{\sharp\dagger}"] \arrow[dl, "x^\sharp"] & \nu H \arrow[d, "\tau"] \\
HTX \arrow[rr, "Hx^{\sharp\dagger}"] & & H\nu H
\end{tikzcd}$$

With that, [SBBR13] turned a HT-coalgebra x into an H-coalgebra $x^\sharp$, giving semantics to the original HT-coalgebra $x^\dagger = x^{\sharp\dagger} \cdot \eta_X$.

This principle can be interpreted as a functor

$$T' : \mathsf{Coalg}(HT) \to \mathsf{Coalg}H^T.$$

T' is defined on objects $X \xrightarrow{x} HTX$ as $TX \xrightarrow{x^\sharp} HTX$ and on morphisms

$$(f : (X, x) \to (Y, y)) \mapsto (Tf : (TX, x^\sharp) \to (TY, y^\sharp)).$$

One can easily prove that Tf is indeed a H^T-coalgebra homomorphism, see [BMS13, Proof of Lemma 3.27], hence T' is a functor.

Example 5.19. Note that the determinization depends heavily on the choice of the lifting H^T. Consider $H = \mathbb{N} \times (-)$ and $T = (-)^+$, i.e. the monad for semigroups, which maps each set X to the set of non-empty words over X. This is an interesting instance of our framework, because there is a finitely generated semigroup that is not finitely presentable, see [CRRT96, Example 4.5]. The final H-coalgebra is carried by streams of natural numbers $\mathbb{N}^\omega$. Consider the coalgebra carried by a two element set $X = \{o, y\}$

$$o \mapsto (0, yo), \quad y \mapsto (1, o).$$

1. For the lifting, where the semigroup structure on $\mathbb{N}$ is that of usual addition of natural numbers, $o \in X^+$ induces the following stream:

$$\begin{array}{lcccccc} \text{State from } X^+: & o & yo & oyo & yooyo & oyoyooyo & yooyooyoyooyo \\ \text{Observation:} & 0 & 1 & 1 & 2 & 3 & 5 \end{array} \cdots$$

   So $x^{\sharp\dagger}(o) \in \mathbb{N}^\omega$ is the stream of Fibonacci numbers

2. Alternatively, equip $\mathbb{N}$ with bitwise XOR (exclusive or), i.e. $n \oplus m$ is the natural number obtained by performing digit-wise XOR on the binary representations of n and m (like the operator ^ in the C programming language). XOR is commutative, 0 is its unit and each n is self-inverse.

$$\begin{array}{lcccccc} \text{State from } X^+: & o & yo & oyo & yooyo & oyoyooyo & yooyooyoyooyo \\ \text{Observation:} & 0 & 1 & 1 & 1^2 = 0 & 1^3 = 1 & 1^5 = 1 \end{array} \cdots$$

   The stream of observations are the Fibonacci numbers modulo 2.

So we get completely different determinization depending on the choice of H^T.

Lemma 5.20 ([BMS13, Lemma 3.24]). *Every fixpoint (C, c) of HT carries a unique T-algebra $\gamma : TC \to C$ structure such that $c : C \to HTC$ is a T-algebra homomorphism. The T-algebra structure γ*

$$\begin{tikzcd} TC \arrow[rr, "c^\sharp"] \arrow[d, "\gamma"'] & & HTC \arrow[d, "H\gamma"] \\ C \arrow[r, "c"] & HTC \arrow[r, "H\gamma"] & HC \end{tikzcd}$$

is a H^T-coalgebra homomorphism.

For the fixpoint $(\vartheta(HT), r : \vartheta(HT) \to HT\vartheta(HT))$, this implies a lifting to $\mathsf{Coalg}H^T$:

$$\vartheta(HT) \xrightarrow{r} HT\vartheta(HT) \xrightarrow{H\gamma} H\vartheta(HT).$$

Lemma 5.21 ([BMS13, Lemma 3.39]). *For any $x^\sharp : TX \to HTX$, with X finite, in $\mathsf{Coalg}H^T$, there is a canonical H^T-coalgebra homomorphism into $(\vartheta(HT), H\gamma \cdot r)$.*

Proof. For the corresponding $x : X \to HTX$, there is a unique HT-coalgebra homomorphism

$$x^\dagger : (X, x) \to (\vartheta(HT), r),$$

which is mapped to the H^T-coalgebra homomorphism $Tx^\dagger$.

$$\begin{tikzcd} TX \arrow[r, "x^\sharp"] \arrow[d, "Tx^\dagger"'] & HTX \arrow[d, "HTx^\dagger"] \\ T\vartheta(HT) \arrow[r, "r^\sharp"] & HT\vartheta(HT) \end{tikzcd}$$

Composing $Tx^\dagger$ with γ from Lemma 5.20 for $C = \vartheta(HT)$, gives a H^T-coalgebra homomorphism from $(TX, x^\sharp)$ to $\vartheta(HT) \xrightarrow{H\gamma \cdot r} H\vartheta(HT)$. $\square$

Remark 5.22. Bonsangue, Milius, and Silva [BMS13] consider the case of finitely presentable objects and lfp coalgebras. Under their assumptions, they consider the rational fixpoint of the Set-functor HT and show that its image in νH is the rational fixpoint of the lifted H. There, a crucial argument is, that they have the previous Lemma 5.21 extended to any f.p. carried H^T-coalgebra whereas we have it only for coalgebras of the form $x^\sharp : TX \to HTX$ with X a finite set.

So we need a different approach to define the image of all the coalgebras from $\mathsf{Coalg}_{\mathrm{fg}}HT$ in νH.

5.3.2 How the LFF helps

Let us now see, how the LFF of a lifting H^T looks like. For each finite set X, $FX \in \mathsf{Set}^T$ is finitely presentable. Thus, for each $x : X \to HTX$ with finite X, the coalgebra $x^\sharp : FX \to H^T FX$ is in $\mathsf{Coalg}_{\mathrm{fg}}H^T$.

Further, each finitely generated Y is the strong quotient of some FX and we can derive a similar property for coalgebras:

Proposition 5.23. *Every f.g. carried H^T-coalgebra $y : Y \to H^T Y$ is the strong quotient of a H^T coalgebra $x : FX \to H^T FX$ with X finite.*

Proof. The object Y is a strong quotient of a free object $q : FX \to Y$, with X finite. Let ϵ be the counit of the adjunction, i.e. we have $\epsilon_Y : FUY \to Y$. By the projectivity of FUY, we get a $p : FUY \to FX$ with $q \cdot p = \epsilon_Y$. On the other hand, the universal property of the adjunction gives us a unique $\bar{q} : X \to UY$ with $q = \epsilon_Y \cdot \bar{q}$, in a picture:

$$\begin{tikzcd} FX \arrow[r, "F\bar{q}"] \arrow[dr, two heads, "q"'] & FUY \arrow[r, "p"] \arrow[d, "\epsilon_Y"] & FX \arrow[dl, two heads, "q"] \\ & Y & \end{tikzcd} \tag{5.9}$$

Furthermore, we can define a coalgebra structure on FUY by the function

$$UY \xrightarrow{Uy} UH^T Y = HUY \xrightarrow{H\eta_{UY}} HUFUY = UH^T FUY,$$

which induces a $y^T : FUY \to H^T FUY$. With that, ϵ_Y becomes an H^T-coalgebra homomorphism $(FUY, y^T) \to (Y, y)$. This can be shown by using the universal property of FUY:

$$\begin{tikzcd} UY \arrow[r, "Uy"] \arrow[d, "\mathrm{id}_{UY}"'] & UH^T Y = HUY \arrow[r, "H\eta_{UY}"] \arrow[dr, equal] & UH^T FUY \arrow[d, "HU\epsilon_Y"] \\ UY \arrow[rr, "Uy"'] & & UH^T Y = HUY \end{tikzcd}$$

All parts commute, hence $y \cdot \epsilon_Y = H^T \epsilon_Y \cdot y^T$ and ϵ_Y is a coalgebra homomorphism. Composing this with the triangles from (5.9) for p and $F\bar{q}$ gives that q is a coalgebra homomorphism:

$$\begin{tikzcd} FX \arrow[r, "F\bar{q}"] \arrow[dr, two heads, "q"'] & FUY \arrow[r, "y^T"] \arrow[d, "\epsilon_Y"] & H^T FUY \arrow[r, "H^T p"] \arrow[d, "H^T \epsilon_Y"'] & H^T FX \arrow[dl, "H^T q"] \\ & Y \arrow[r, "y"'] & H^T Y & \end{tikzcd}$$

$\square$

What can we say about the locally finite fixpoint of H^T in Set^T? First, one can follow from [Bar04, Theorem 3.2.3] and [PT97], that νH lifts to a final coalgebra of H^T, see [BMS13, Notation 3.22]. For that, we only need to give a T-algebra structure to νH:

$$\begin{tikzcd}
T(\nu H) \arrow[r, "T\tau"] \arrow[d, dashed, "\alpha"] & TH(\nu H) \arrow[r, "\lambda_{\nu H}"] & HT(\nu H) \arrow[d, "H\alpha"] \\
\nu H \arrow[rr, "\tau"] & & H(\nu H)
\end{tikzcd}$$

Note that, the right-adjoint U preserves monos and is faithful. So the monomorphisms in Set^T are precisely the mono-carried algebra homomorphisms. This is important, because with that, H^T preserves monos as well: for any mono m in $\mathcal{B}^T$, $HUm = UH^Tm$ is a mono in $\mathcal{B}$ and H^Tm is a mono as well.

As T is finitary, filtered colimits in Set^T are built as in Set and H^T is also finitary. In total, everything from Assumption 4.21 is met for $H^T : \mathcal{B}^T \to \mathcal{B}^T$. Hence, we get its locally finite fixpoint

$$\vartheta H^T \xrightarrow{\ \ell\ } H^T\vartheta H^T.$$

In the remainder of this section we want to show, that this is indeed the desired object if one wants to talk about the semantics of HT-coalgebras. We can show this by proving that $U\vartheta H^T$ is precisely the collection of images of all the determinized HT-coalgebras in νH.

By "the collection of images" we mean the filtered colimit of the determinization functor

$$D \equiv \big(\mathsf{Coalg}_{\mathrm{fg}}HT \xrightarrow{T'} \mathsf{Coalg}H^T \xrightarrow{U} \mathsf{Coalg}H.\big)$$

$\mathsf{Coalg}H$ is cocomplete and $\mathsf{Coalg}_{\mathrm{fg}}HT$ essentially small, so the filtered colimit

$$\operatorname{colim} D =: (K, k : K \to HK) \text{ with injections } \mathsf{in}_X : (TX, x^\sharp) \to (K, k)$$

exists. The image of the unique homomorphism $k^\dagger : (K, k) \to (\nu H, \tau)$ is obtained by the (strong epi carried,mono-carried)-factorization as in Proposition 4.20.

$$\begin{tikzcd}
(K,k) \arrow[r, two heads] \arrow[rr, bend right, "k^\dagger"'] & (I,i) \arrow[r, tail] & (\nu H, \tau)
\end{tikzcd}$$

Proposition 5.24. *The intermediate coalgebra (I, i) is precisely the locally finite fixpoint of the lifted H^T, i.e. $(I, i) \cong (U\vartheta H^T, U\ell)$.*

Proof. First of all, $(\vartheta H^T, \ell)$ is final for all $(TX, x^\sharp)$, with X finite, so it is a competing cocone for (K, k):

$$\begin{tikzcd}
(TX, x^\sharp) \arrow[d, "\mathsf{in}_X"'] \arrow[dr, "Ux^{\sharp\dagger}"] & \\
(K,k) \arrow[r, dashed, "w"'] \arrow[d, two heads, "e"'] & (U\vartheta H^T, U\ell) \arrow[d, tail, "n"] \\
(I,i) \arrow[r, tail, "m"'] & (\nu H, \tau)
\end{tikzcd}$$

Hence, w is induced making the triangle commute. Any (G, g) in $\mathsf{Coalg}_{\mathrm{fg}}H^T$ is the quotient of some $(TX, x^\sharp)$. And on the other hand, the $g^\dagger : (G, g) \to (\vartheta H^T, \ell)$ are jointly epic. Hence, the $x^{\sharp\dagger}$ are jointly epic as well, and by Proposition 5.7 the $Ux^{\sharp\dagger}$, too. So by Example 3.4, the function w is epic, and – as we are in Set – even a strong epimorphism. In other words, $(U\vartheta H^T, U\ell)$ is the (unique) image of (K, k) in $(\nu H, \tau)$. ☐

Corollary 5.25. *By Lemma 4.32, the intermediate object (I, i), and therefore also $U\vartheta H^T$, is precisely the union of images of the determinized HT-coalgebras:*

$$U\vartheta H^T = I = \bigcup_{\substack{x: X \to HTX \\ X \text{ finite}}} x^\dagger[TX] \subseteq \nu H. \tag{5.10}$$

5.3.3 Context-free Languages

There are several characterizations of context-free languages. One is by *context-free grammars*, another is by *non-deterministic stack machines*. Both approaches have been characterized coalgebraically recently. After recalling each, we will apply the LFF using (5.10) to obtain the context-free languages as a subcoalgebra of the final coalgebra for the language functor $2 \times (-)^\Sigma$.

Context-free Grammars

A grammar is called context-free, if the left-hand side of each production consists of a single non-terminal symbol from X and each right-hand side consists of a finite sequence of non-terminals from X and terminal symbols from Σ.

Greibach [Gre65] proved, that every context-free grammar can be turned into a normal-form, for which the right-hand side is either empty or consists of a single terminal-symbol followed by a finite sequence of non-terminals. Winter, Bonsangue, and Rutten [WBR13] turned grammars in Greibach normal-form into coalgebras for the functor

$$2 \times \mathcal{P}_\mathsf{f}((-)^*)^\Sigma = H \circ T, \qquad \text{for } H = 2 \times (-)^\Sigma \text{ and } T = \mathcal{P}_\mathsf{f}((-)^*).$$

More precisely, for each grammar in Greibach normal-form with non-terminals X they define an HT-coalgebra $\langle o, \delta \rangle : X \to 2 \times TX^\Sigma$ by

$$o(x) = \begin{cases} 1 & \text{if } x ::= \varepsilon \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \delta(x, a) = x_a = \{w \in X^* \mid x ::= aw\}$$

where ε denotes the empty word. The considered monad $T = \mathcal{P}_\mathsf{f}((- + \Sigma)^*)$ is obtained by composing the monad for *idempotent semi-rings* $T' = \mathcal{P}_\mathsf{f}((-)^*)$ with the exception monad $(- + \Sigma)$. As Hyland, Plotkin, and Power [HPP06, Corollary 1] observed, composition with the exception monad maps a monad T' to a monad T again, and an Eilenberg-Moore algebra on X for T' is just an Eilenberg-Moore algebra $TX \to X$ together with a pointing $\Sigma \to X$.

Winter, Bonsangue, and Rutten [WBR13, page 30] define a T-algebra structure on HTX, which can be generalized to a lifting. Consider the free idempotent semi-ring $(\mathcal{P}_\mathsf{f}((X+\Sigma)^*), \cdot, +, 1_{TX}, 0_{TX})$ on $X + \Sigma$ with the obvious pointing $p_{TX} = \eta^T_X = \eta^{T'}_{X+\Sigma} \cdot \mathsf{inr} : \Sigma \to \mathcal{P}_\mathsf{f}((X + \Sigma)^*)$, the obvious inclusion $i_{TX} : 2 \hookrightarrow TX$, and note that $1_{TX} = \{\varepsilon\}$ and $0_{TX} = \emptyset$. Then define an T'-algebra structure on $2 \times TX^\Sigma$ by:

Connective	in 2	in TX^Σ
0	0_B	$a \mapsto 0_{TX}$
1	1_B	$a \mapsto 0_{TX}$
$(o_1, \delta_1) \oplus (o_2, \delta_2)$	$o_1 \vee o_2$	$a \mapsto \delta_1(a) + \delta_2(a)$
$(o_1, \delta_1) \otimes (o_2, \delta_2)$	$o_1 \wedge o_2$	$a \mapsto \delta_1(a) \cdot i_{TX}(o_2) + \delta_1(a) \cdot \sum_{b \in \Sigma}(p(b) \cdot \delta_2(b)) + i_{TX}(o_1) \cdot \delta_2(a)$
$b \in \Sigma$	1_B	$a \mapsto \begin{cases} 0_{TX} & b \neq a \\ 1_{TX} & b = a \end{cases}$

This is just the definition of [WBR13, page 30] and by [WBR13, Proposition 7.1], the first four lines define an idempotent semi-ring, and the last line a pointing $\Sigma \to 2 \times TX^{\Sigma}$.

Lemma 5.26. *For any T-algebra homomorphism $f : TX \to TY$, we have a T-algebra homomorphism $Hf = 2 \times f^{\Sigma} : 2 \times TX^{\Sigma} \to 2 \times TY^{\Sigma}$.*

Proof. The operations on $2 \times \mathcal{P}_{\mathsf{f}}((X+\Sigma)^*)^{\Sigma}$ and $2 \times \mathcal{P}_{\mathsf{f}}((X+\Sigma)^*)^{\Sigma}$ are defined as compositions of operations from TX and TY respectively, so $2 \times f^{\Sigma}$ preserves the structure trivially:

$$\begin{aligned}
Hf(0_B, a \mapsto 0_{TX}) &= (0_B, a \mapsto f(0_{TX})) = (0_B, a \mapsto 0_{TY}) \text{ (analogously for 1 and the pointing)}\\
Hf\big((o_1,\delta_1) \oplus (o_2,\delta_2)\big) &= (o_1 \vee o_2, a \mapsto f(\delta_1(a) + \delta_2(b)))\\
&= (o_1, a \mapsto f(\delta_1(a))) \oplus (o_1, a \mapsto f(\delta_1(a))) = Hf(o_1,\delta_1) \oplus Hf(o_2,\delta_2)\\
Hf\big((o_1,\delta_1) \otimes (o_2,\delta_2)\big) &= Hf\big(o_1 \wedge o_2, a \mapsto \delta_1(a) \cdot i_{TX}(o_2) + \delta_1(a) \cdot \sum_{b\in\Sigma} (p_{TX}(b) \cdot \delta_2(b)) + i_{TX}(o_1) \cdot \delta_2(a)\big)\\
&= \big(o_1 \wedge o_2, a \mapsto \ f(\delta_1(a)) \cdot f(i_{TX}(o_2)) + f(\delta_1(a)) \cdot \sum_{b\in\Sigma} (f(p_{TX}(b)) \cdot f(\delta_2(b)))\\
&\qquad + f(i_{TX}(o_1)) \cdot f(\delta_2(a))\big)\\
&= (o_1, a \mapsto f(\delta_1)) \otimes (o_2, a \mapsto f(\delta_2)) = Hf(o_1,\delta_1) \otimes Hf(o_2,\delta_2).
\end{aligned}$$

For the last equality, note that $f \circ p_{TX} = p_{TY}$ because f preserves the Σ-pointing and $f \circ i_{TX} = i_{TY}$ because f preserves both 0 and 1. ☐

In total, we get a lifting $\hat{H} : \mathsf{Set}_T \to \mathsf{Set}^T$ of $H = 2 \times (-)^{\Sigma}$, which extends to a unique lifting $H^T : \mathsf{Set}^T \to \mathsf{Set}^T$ by Theorem 5.18. This allows applying the technique of generalized determinization and one concludes:

Theorem 5.27 ([WBR13, Theorem 3.8])**.** *The following conditions are equivalent:*

1. *A language L is context-free.*
2. *There exists a coalgebra $x : X \to HTX$, X finite, with its determinization $x^{\sharp} : TX \to HTX$ and an $s \in X$ such that*

$$\begin{array}{ccccc}
x & \longmapsto & \{x\} & \longmapsto & L\\
\cap & & \cap & & \cap\\
X & \xrightarrow{\eta^T_X} & \mathcal{P}_{\mathsf{f}}((X+\Sigma)^*) & \xrightarrow{x^{\sharp\dagger}} & \nu H.
\end{array}$$

Combining this result with (5.10) gives:

Corollary 5.28. *The locally finite fixpoint of H^T is carried by the context-free languages on Σ.*

Stack Machines

A characterization of context-free languages in terms of automata is by stack machines, i.e. by automata equipped with a stack:

Definition 5.29 (Stack monad, [Gon13, Proposition 5])**.** For a finite set of stack symbols Γ, the *stack monad* is the submonad T of the store monad $(- \times \Gamma^*)^{\Gamma^*}$ for which the elements $\langle r, t\rangle$ of $TX \subseteq (X \times \Gamma^*)^{\Gamma^*}$ satisfy the following restriction: there exists k depending on r, t such that for every $w \in \Gamma^k$ and $u \in \Gamma^*$, $r(wu) = r(w)$ and $t(wu) = t(w)$.

Intuitively, this means that we cannot access (or depend our computation on) the entire stack but only the k topmost elements. The corresponding theory consists of operations push and pop with the intuitive equations, see [Gon13, Proposition 5] for the proof of the correspondence. All the operations have finite arities, so the stack monad is finitary, by Remark 5.4.

Here, the functor $HX = B \times X^{\Sigma}$ is considered, where

$$B = \{p \in 2^{\Gamma^*} \mid \exists k \in \mathbb{N}_0 : \forall w, u \in \Gamma^*, |w| \geq k : p(wu) = p(w)\} \subseteq 2^{\Gamma^*}$$

is the set of all predicates mapping initial stack configurations to output values from 2 taking only the topmost k elements of the stack into account. One can easily show that the final H-coalgebra is carried by B^{Σ^*}. But this carrier can be considered as functions assigning initial stack-configurations to formal languages, because

$$B^{\Sigma^*} \cong (2^{\Gamma^*})^{\Sigma^*} \cong 2^{\Gamma^* \times \Sigma^*} \cong (2^{\Sigma^*})^{\Gamma^*} \cong \mathcal{P}(\Sigma^*)^{\Gamma^*}. \tag{5.11}$$

From now on, we will consider $\mathcal{P}_{\mathsf{f}}(\Sigma^*)^{\Gamma^*}$ as the carrier of νH.

The functor $HX = B \times X^{\Sigma}$ lifts to Set^T and T-automata are precisely the H^T-coalgebras carried by a TX with X finite, see [GMS14, Remark 4.2, Remark 4.5].

Theorem 5.30 ([GMS14, Theorem 5.5])**.** *The following conditions are equivalent:*

1. *A language L is a real-time deterministic context-free language.*
2. *There exists a coalgebra $x : X \to HTX$, X finite, with its determinization $x^\sharp : TX \to HTX$ and there exist $s \in X$ and $\gamma_0 \in \Gamma$ such that*

$$\begin{tikzcd} x \arrow[rrr, mapsto] & & & L \\[-20pt] \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\[-20pt] X \arrow[r, "\eta^T_X"] & TX \arrow[r, "x^{\sharp\dagger}"] & \mathcal{P}(\Sigma^*)^{\Gamma^*} \arrow[r, "\mathsf{ev}_{\gamma_0}"] & \mathcal{P}(\Sigma^*), \end{tikzcd}$$

   where $\mathsf{ev}_y : Z^Y \to Z$ denotes the evaluation of a function $Y \to Z$ at $y \in Y$.

Corollary 5.31. *The locally finite fixpoint of H^T is carried by maps $U\vartheta H^T \subseteq \mathcal{P}_{\mathsf{f}}(\Sigma^*)^{\Gamma^*}$ whose image*

$$\{f(\gamma_0) \mid U\vartheta H^T \ni f : \Gamma^* \to \mathcal{P}_{\mathsf{f}}(\Sigma^*), \gamma_0 \in \Gamma\} \subseteq \mathcal{P}_{\mathsf{f}}(\Sigma^*)$$

is precisely the class of real-time deterministic context-free languages over Σ.

In [GMS14, Section 6] the stack monad is extended to a non-deterministic stack monad S, i.e. S denotes a submonad of the non-deterministic store monad $\mathcal{P}_{\mathsf{f}}(- \times \Gamma^*)^{\Gamma^*}$.

Theorem 5.32 ([GMS14, Theorem 6.5])**.** *The following conditions are equivalent:*

1. *A language L is context-free language.*
2. *There exists a coalgebra $x : X \to HSX$, X finite, with its determinization $x^\sharp : SX \to HSX$ and there exist $s \in X$ and $\gamma_0 \in \Gamma$ such that*

$$\begin{tikzcd} x \arrow[rrr, mapsto] & & & L \\[-20pt] \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\[-20pt] X \arrow[r, "\eta^S_X"] & SX \arrow[r, "x^{\sharp\dagger}"] & \mathcal{P}(\Sigma^*)^{\Gamma^*} \arrow[r, "\mathsf{ev}_{\gamma_0}"] & \mathcal{P}(\Sigma^*). \end{tikzcd}$$

These two results can be combined with (5.10):

Corollary 5.33. *The locally finite fixpoint of H^S is carried by maps $U\vartheta H^S \subseteq \mathcal{P}_{\mathsf{f}}(\Sigma^*)^{\Gamma^*}$ whose image*

$$\{f(\gamma_0) \mid U\vartheta H^S \ni f : \Gamma^* \to \mathcal{P}_{\mathsf{f}}(\Sigma^*), \gamma_0 \in \Gamma\} \subseteq \mathcal{P}_{\mathsf{f}}(\Sigma^*)$$

is precisely the class of context-free languages over Σ.

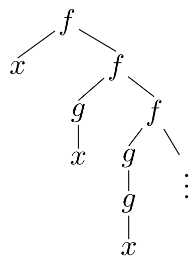
5.4 Algebraic Trees

Given a signature Σ, a *recursive program scheme* (or *rps* for short) defines operations $\varphi_1, \dots, \varphi_k$ with arities $n_1, \dots, n_k$ recursively. Such an operation φ_i of arity n_i is defined by a term, built from symbols from Σ, operations $\varphi_1, \dots, \varphi_k$, and n_i variables $x_1, \dots, x_{n_i}$. A solution of an rps are k possibly infinite trees $t_1, \dots, t_n$ over Σ and over the variables, where tree t_i solves φ_i.

As an example, consider the signature $\Sigma = \{f/2, g/1\}$ and the rps

$$\varphi(x) = f(x, \varphi(g(x))).$$

The solution of φ is this tree:



Generally, *algebraic trees* are defined to be those trees that are solutions of recursive program schemes.

When generalizing from signatures to finitary endofunctors $H : \mathcal{B} \to \mathcal{B}$ on an lfp category, [AMV11a] gives a categorical description of the algebraic trees as the *context-free monad*, C^H. It is obtained as a quotient of the *second-order iterative monad*, S^H.

The construction is done on a kind of slice category $H/\mathsf{Mnd_f}(\mathsf{Set})$, in which objects are finitary monads M together with a natural transformation $H \to M$.

5.4.1 Why the help of the LFF is needed

Before defining the details, let us show, that it is very likely that we need the LFF and that the rational fixpoint is not sufficient. That is because in $\mathsf{Mnd_f}(\mathsf{Set})$ – and thus also in $H/\mathsf{Mnd_f}(\mathsf{Set})$ – there are finitely generated objects that are not finitely presentable. The main role in this observation is the monad for the free monoid action.

Definition 5.34 (Free monoid action)**.** Let $(M, e, \cdot)$ be a monoid. Define the (obviously finitary) Set-endofunctor $T_M X = M \times X$ with the unit

$$\eta_X : X \to M \times X, \quad x \mapsto (e, x)$$

and the multiplication

$$\mu_X : M \times M \times X \to M \times X, \quad (m_1, m_2, x) \mapsto (m_1 \cdot m_2, x).$$

The monad laws for associativity and the unit are directly inherited from the respective monoid laws.

We need the definition monoidal only for the following Proposition 5.40. For more details see [Kel82, Chapter 1].

Definition 5.35. A *monoidal category* $(\mathcal{C}, \otimes, I)$ consists of a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and natural isomorphisms

- $a_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$

- $\lambda_X : I \otimes X \to X$, $\lambda'_X : X \otimes I \to X$

fulfilling the identities

$$\begin{tikzcd}
 & X\otimes(Y\otimes(Z\otimes T)) \arrow[dl, "I\otimes a"'] \arrow[dr, "a"] & \\
X\otimes((Y\otimes Z)\otimes T) \arrow[d, "a"] & & (X\otimes Y)\otimes(Z\otimes T) \arrow[d, "a"] \\
(X\otimes(Y\otimes Z))\otimes T \arrow[rr, "a\otimes I"] & & ((X\otimes Y)\otimes Z)\otimes T
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes(I\otimes Y) \arrow[rr, "a"] \arrow[dr, "X\otimes\lambda_Y"'] & & (X\otimes I)\otimes Y \arrow[dl, "\lambda'_X\otimes Y"] \\
 & X\otimes Y &
\end{tikzcd}$$

Definition 5.36. A monoid in a monoidal category $(\mathcal{C}, \otimes, I)$ is an object M together with natural transformations $m : M \otimes M \to M$ and $u : I \to M$ such that we have:

$$\begin{tikzcd}
M\otimes I \arrow[r, "M\otimes u"] \arrow[dr, "\lambda'_M"'] & M\otimes M \arrow[d, "m"] & I\otimes M \arrow[l, "M\otimes u"'] \arrow[dl, "\lambda_M"] \\
 & M &
\end{tikzcd}
\quad
\begin{tikzcd}
(M\otimes M)\otimes M \arrow[r, "a_{M,M,M}"] \arrow[d, "m\otimes M"'] & M\otimes(M\otimes M) \arrow[r, "M\otimes m"] & M\otimes M \arrow[d, "m"] \\
M\otimes M \arrow[rr, "m"] & & M
\end{tikzcd}$$

Example 5.37.
- Monoids in $(\mathsf{Set}, \times, 1)$ are monoids in the ordinary sense.
- Monoids in the (finitary) functor category $(\mathsf{Fun_f}(\mathcal{C}), \circ, \mathrm{Id})$, with composition as the tensor, are (finitary) monads on $\mathcal{C}$.

Definition 5.38 (Strong monad)**.** A monad (T, η, μ) on a monoidal category $(\mathcal{C}, \otimes, I)$ is called *strong* if it is equipped with a natural transformation $t_{X,Y} : TX \otimes Y \to T(X \otimes Y)$, called *strength*, such that the following diagrams commute:

$$\begin{tikzcd}
TX\otimes I \arrow[r, "t_{X,I}"] \arrow[dr, "\lambda'_{TX}"'] & T(X\otimes I) \arrow[d, "T\lambda'_X"] \\
 & TX
\end{tikzcd}
\quad
\begin{tikzcd}
 & X\otimes Y \arrow[dl, "\eta_X\otimes Y"'] \arrow[dr, "\eta_{X\otimes Y}"] & \\
TX\otimes Y \arrow[rr, "T_{X,Y}"] & & T(X\otimes Y) \\
T^2X\otimes Y \arrow[u, "\mu_X\otimes Y"] \arrow[r, "t_{TX,Y}"] & T(TX\otimes Y) \arrow[r, "Tt_{X,Y}"] & T^2(X\otimes Y) \arrow[u, "\mu_{X\otimes Y}"']
\end{tikzcd} \tag{5.12}$$

$$\begin{tikzcd}
TX\otimes(Y\otimes Z) \arrow[rr, "t_{X,Y\otimes Z}"] \arrow[d, "a_{TX,Y,Z}"'] & & T(X\otimes(Y\otimes Z)) \arrow[d, "Ta_{X,Y,Z}"] \\
(TX\otimes Y)\otimes Z \arrow[r, "t_{X,Y}\otimes Z"] & T(X\otimes Y)\otimes Z \arrow[r, "t_{X\otimes Y,Z}"] & T((X\otimes Y)\otimes Z)
\end{tikzcd} \tag{5.13}$$

Example 5.39. All monads on Set are strong. It is not hard to prove that for a monad T on Set, its strength $t_{X,Y} : TX \times Y \to T(X \times Y)$ is given by

$$\begin{tikzcd}
TX\times Y \arrow[r, dashed, "t_{X,Y}"] & T(X\times Y) \\
TX\times 1 \arrow[u, "TX\times y"] \arrow[r, phantom, "\cong"] & T(X\times 1) \arrow[u, "T(X\times y)"']
\end{tikzcd}$$

using the universal property of the coproduct $(TX \times y : TX \times 1 \to TX \times Y)_{y \in Y}$.

Proposition 5.40. *If (T, η, μ) is a strong monad on a monoidal category $(\mathcal{C}, \otimes, I)$, then TI carries a canonical monoid structure*

$$m \equiv \left(TI \otimes TI \xrightarrow{t_{I,TI}} T(I \otimes TI) \xrightarrow{\lambda_{TI}} T^2 I \xrightarrow{\mu_I} TI \right)$$

with the unit $\eta_I : I \to TI$.

This is a commonly known result, see for example [AMV11b, Corollary 4.13] for a proof for point-strength (a slightly weaker notion than strength) and strict monoidal categories (where λ, λ', a are families of identities). For the convenience of the reader we present a detailed proof.

Proof. To see that η_I is (both the left and right) unit of m, consider the following commuting diagram:

$$\begin{tikzcd}
I \otimes TI \arrow[r, "\eta_I \otimes TI"] \arrow[dr, "\eta_{I \otimes TI}"'] \arrow[dd, "\lambda_{TI}"'] \arrow[dr, phantom, "(5.12)", near start, shift left=3] & TI \otimes TI \arrow[d, "t_{I,TI}"] \arrow[r, phantom, "\substack{\text{naturality} \\ \text{of } t_{I,(-)} \text{ for } \eta_I}"] & TI \otimes I \arrow[l, "TI \otimes \eta_I"'] \arrow[d, "t_{I,I}"] \\
 & T(I \otimes TI) \arrow[d, "T\lambda_{TI}"] \arrow[r, phantom, "\substack{\text{naturality} \\ \text{of } T\lambda \text{ for } \eta_I}"] \arrow[l, phantom, "\substack{\text{naturality} \\ \text{of } \eta \text{ for } \lambda_{TI}}"] & T(I \otimes I) \arrow[l, "T(I \otimes \eta_I)"'] \arrow[d, "T\lambda_I"] \\
TI \arrow[r, "\eta_{TI}"] \arrow[dr, equal] & TTI \arrow[d, "\mu_I"] & TI \arrow[l, "T\eta_I"'] \arrow[dl, equal] \\
 & TI &
\end{tikzcd}$$

The proof for the associativity of m can be found in Figure 5.1 on page 62. The referenced Kelly-Lemma says that $\lambda_{A \otimes B} = (\lambda_A \otimes B) \cdot a_{I,A,B}$, see [Kel64, Condition (5)]. $\square$

Remark 5.41. For the free monoid action T_M we have: $T_{(M,\cdot,e)}1 = (M, \cdot, e)$, i.e. exactly what we have put in.

Definition 5.42 (Strong monad morphism)**.** A monad morphism $\sigma : T \to S$ between strong monads T and S is called *strong* provided the commutativity of:

$$\begin{tikzcd}
TX \otimes Y \arrow[r, "t^T_{X,Y}"] \arrow[d, "\sigma_X \otimes Y"'] & T(X \otimes Y) \arrow[d, "\sigma_{X \otimes Y}"] \\
SX \otimes Y \arrow[r, "t^S_{X,Y}"] & S(X \otimes Y)
\end{tikzcd} \tag{5.14}$$

Proposition 5.43. *Any monad morphism $\sigma : T \to S$ between strong monads on* Set *is strong.*

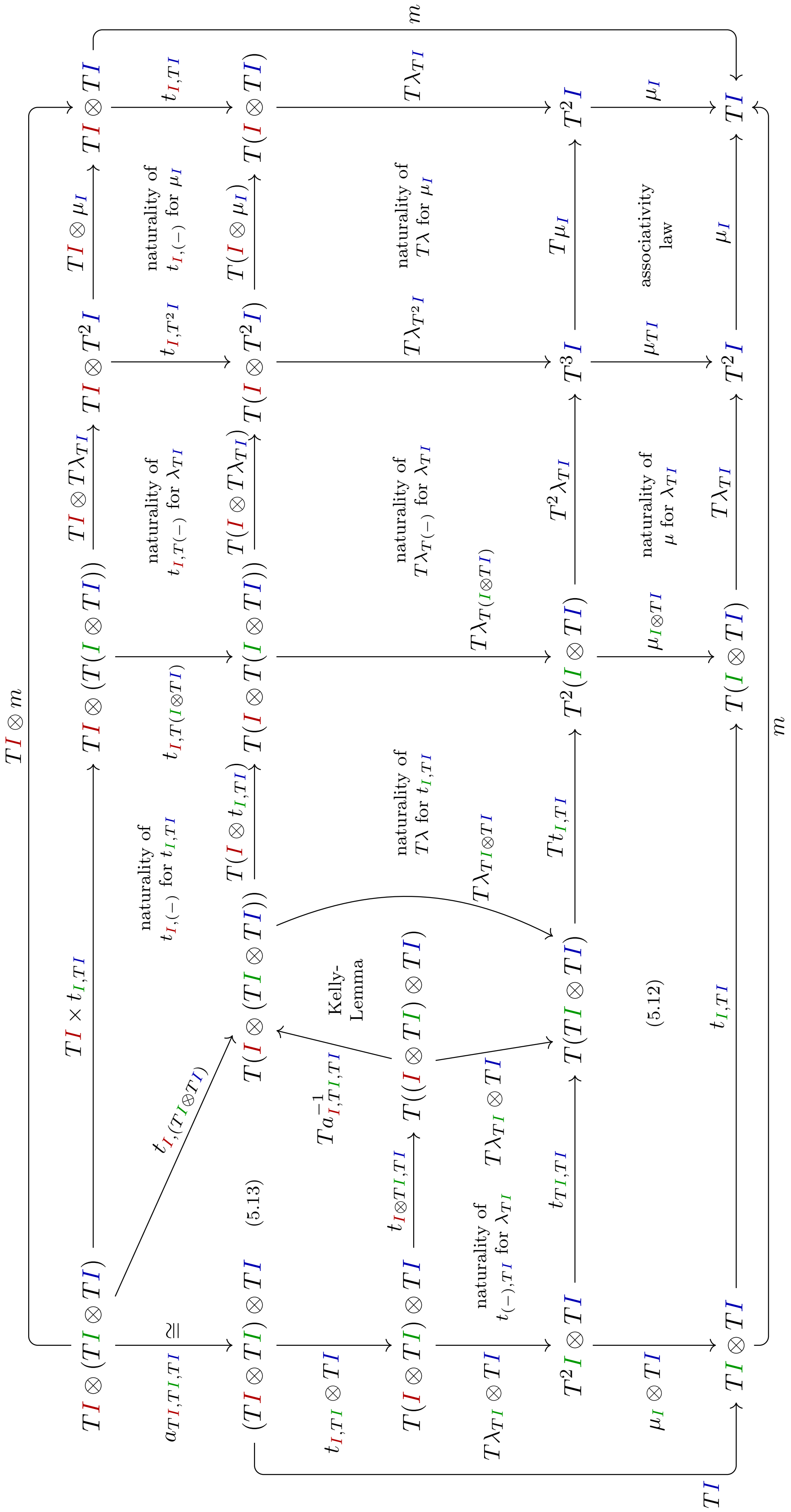


Figure 5.1: Associativity of the monoid TI

Proof. For an arbitrary $y : 1 \to Y$ consider:

$$
\begin{tikzcd}
TX \times Y \arrow[rrrr, "t^T_{X,Y}"] \arrow[rrrr, phantom, "\text{Naturality of } t^T_{X,(-)} \text{ for } y"', shift right=3] \arrow[ddddd, "\sigma_X \times Y"'] & & & & T(X \times Y) \arrow[ddddd, "\sigma_{X \times Y}"] \\
& TX \times 1 \arrow[ul, "TX \times y"'] \arrow[rr, bend left, "t^T_{X,1}"] \arrow[rr, phantom, "(5.12)"] \arrow[dr, "\lambda'_{TX}"'] \arrow[ddd, "\sigma_X \times 1"'] \arrow[ddd, phantom, "\text{Naturality of } \sigma", shift right=12] & & T(X \times 1) \arrow[ur, "T(X \times y)"] \arrow[ddd, "\sigma_{X \times 1}"] \arrow[ddd, phantom, "\text{Naturality of } \sigma"', shift left=12] & \\
& & TX \arrow[ur, "T\lambda'^{-1}_X"'] \arrow[d, "\sigma_X"] & & \\
& & SX \arrow[dr, "S\lambda'^{-1}_X"] & & \\
& SX \times 1 \arrow[ur, "\lambda'_{SX}"] \arrow[rr, bend right, "t^S_{X,1}"'] \arrow[rr, phantom, "(5.12)"] \arrow[dl, "SX \times y"] & & S(X \times 1) \arrow[dr, "S(X \times y)"] & \\
SX \times Y \arrow[rrrr, "t^S_{X,Y}"'] \arrow[rrrr, phantom, "\text{Naturality of } t^S_{X,(-)} \text{ for } y", shift left=3] & & & & S(X \times Y)
\end{tikzcd}
$$

As the $TX \times y : TX \times 1 \to TX \times Y$ are jointly epic, (5.14) is fulfilled. ☐

Lemma 5.44. *For $M \in \mathsf{Set}$ and $S \in \mathsf{Fun_f}(\mathsf{Set})$, we have the isomorphism*

$$\mathsf{Fun_f}(\mathsf{Set})(M \times (-), S) \cong \mathsf{Set}(M, S1).$$

which is natural in S.

This statement is an instance of the Yoneda lemma, see e.g. [Awo10, Lemma 8.2], combined with coproducts.

Proof. By the universal property of the coproduct $M \cong \coprod_{m \in M} 1$, we have a one-to-one correspondence between natural transformations $\alpha : M \times (-) \to S$ and M-indexed families of natural transformations $(\beta^m : \mathrm{Id} \to S)_{m \in M}$. Latter is in bijective correspondence to elements of $S1$ because of the Yoneda lemma for $\mathrm{Id} = \mathsf{Set}(1, -)$. So we have

$$\mathsf{Fun_f}(\mathsf{Set})(M \times (-), S) \cong (S1)^M = \mathsf{Set}(M, S1).$$

Moreover, the Yoneda lemma also tells us that the bijection is as follows: $\alpha : M \times (-) \to S$ corresponds to $\alpha_1 : M \times 1 \to S1$. Conversely, for every $s \in S1$ we get a natural transformation $\beta^s : \mathrm{Id} \to S$ with $\beta^s_X : X \to SX, x \mapsto Sx(s)$, and so by the universal property of the coproduct we get for every $f : M \to S1$ the natural transformation $\alpha = [\beta^{f(m)}]_{m \in M} : M \times (-) \to S$. Again by Yoneda, the isomorphism is natural in S. ☐

Lemma 5.45. *For $M \in \mathsf{Mon}$ and $S \in \mathsf{Mnd_f}(\mathsf{Set})$, we have*

$$\mathsf{Mnd_f}(\mathsf{Set})(T_M, S) \cong \mathsf{Mon}(M, S1),$$

which is natural in S, and in which $S1$ is the monoid as in Proposition 5.40,

Proof. After Lemma 5.44, it is sufficient to show that the bijection maps monoid homomorphisms to monad morphisms and vice versa.

1. Take a monad morphism α. To see that α_1 preserves the monoid multiplication, consider:

$$
\begin{tikzcd}[column sep=large]
(M\times 1)\times(M\times 1) \arrow[r, "\alpha_1\times(M\times 1)"] \arrow[d, "t^M_{1,M\times 1}"'] \arrow[rr, bend left, "\alpha_1\times\alpha_1"] \arrow[dr, phantom, "\substack{\text{Strong monad}\\ \text{morphism}\\ (5.14)}"] & S1\times(M\times 1) \arrow[r, "S1\times\alpha_1"] \arrow[d, "t^S_{1,M\times 1}"] \arrow[dr, phantom, "\substack{\text{nat. of } t^S_{1,(-)}\\ \text{for } \alpha_1}"] & S1\times S1 \arrow[d, "t^S_{1,S1}"] \\
M\times(1\times(M\times 1)) \arrow[r, "\alpha_1\times(M\times 1)"] \arrow[dd, "M\times\lambda_{M\times 1}"'] \arrow[dr, phantom, "\substack{\text{nat. of } \alpha\\ \text{for } \lambda_{M\times 1}}"] & S(1\times(M\times 1)) \arrow[r, "S(1\times\alpha_1)"] \arrow[d, "S\lambda_{M\times 1}"] \arrow[dr, phantom, "\substack{\text{nat. of } S\lambda\\ \text{for } \alpha_1}"] & S(1\times S1) \arrow[dd, "S\lambda_{S1}"] \\
 & S(M\times 1) \arrow[dr, "S\alpha_1"] & \\
M\times(M\times 1) \arrow[ur, "\alpha_{M\times 1}"] \arrow[rr, "(\alpha\ast\alpha)_1"'] \arrow[d, "\mu_1^{M\times(-)}"'] \arrow[drr, phantom, "\text{Monad morphism}"] & & SS1 \arrow[d, "\mu^S_1"] \\
M\times 1 \arrow[rr, "\alpha_1"'] & & S1
\end{tikzcd}
$$

The commutativity implies that α_1 preserves the monoid multiplication. The monoid unit is trivially preserved, because $\alpha_1 \cdot \eta_1^{M\times(-)} = \eta_1^S$. Hence, α_1 is a monoid homomorphism and so also $\alpha_1 \cdot \lambda_M^{-1} : M \to S1$.

2. For the converse direction, consider some monoid homomorphism $f : M \to S1$ and define α as in Lemma 5.44 by $\alpha_1 = f \cdot \lambda_M : M \times 1 \to S1$. Consider for a set X and an arbitrary $x : 1 \to X$:

$$
\begin{tikzcd}
X \arrow[ddd, "\eta_1^{M\times(-)}"'] \arrow[dddrrr, bend left, "\eta^S_1"] & & & \\
 & 1 \arrow[ul, "x"'] \arrow[d, "\eta_1^{M\times(-)}"'] \arrow[dr, "\eta^S_1"] & & \\
 & M\times 1 \arrow[r, "\alpha_1"] \arrow[dl, "M\times x"'] & S1 \arrow[dr, "Sx"] & \\
M\times X \arrow[rrr, "\alpha_X"] & & & SX
\end{tikzcd}
$$

   - The inner triangle commutes because α_1 is a monoid homomorphism.
   - The bottom part commutes because α is natural.
   - the other two parts commute because $\eta^{M\times(-)}$ and η^S are natural transformations.

   As the $x : 1 \to X$ are jointly epic, the outer triangle commutes as well, so α preserves the monad unit.

   Note, that by the strength, we have a natural transformation

$$S\lambda \cdot t_{1,(-)} \cdot (f \times -) : M \times (-) \to S.$$

   As the natural transformation $\alpha : M \times (-) \to S$ was uniquely determined by α_1 in Lemma 5.44, we have

$$
\begin{tikzcd}
M\times X \arrow[r, "f\times X"] \arrow[rrr, bend left, "\alpha_X"] & S1\times X \arrow[r, "t^S_{1,X}"] & S(1\times X) \arrow[r, "S\lambda_X"] & SX.
\end{tikzcd}
\tag{5.15}
$$

With that, we can show that α_1 preserves the monad multiplication μ_1:

$$
\begin{tikzcd}
 & & & S(M\times 1) \arrow[d, "S\lambda_M"] \arrow[ddrr, phantom, "\text{Definition of }\alpha_1"] \arrow[dddr, rounded corners, to path={-- ([xshift=8ex]\tikztostart.east) -| ([xshift=12ex]\tikztotarget.east) \tikztonodes -- (\tikztotarget)}, "S\alpha_1"] & \\
M\times(M\times 1) \arrow[urrr, rounded corners, to path={|- (\tikztotarget) \tikztonodes}, "\alpha_{M\times 1}"] \arrow[r, "M\times\lambda_M"] \arrow[dddd, "\mu_1^{M\times(-)}"'] \arrow[ddddr, phantom, "\text{Definition of }\mu^{M\times(-)}"] & M\times M \arrow[rr, "\alpha_M"] \arrow[ur, phantom, "\text{Definition of }\alpha_{M\times 1}"] \arrow[dr, bend left, "f\times M"'] \arrow[ddd, bend right, "\cdot"] \arrow[dr, phantom, "(5.15)", near end] & & SM \arrow[ddr, rounded corners, to path={-| (\tikztotarget) \tikztonodes}, "Sf"] & \\
 & & S1\times M \arrow[r, "t_{1,M}"] \arrow[d, "S1\times f"'] \arrow[dr, phantom, "\text{naturality of } t^S_{1,(-)} \text{ for } f"] & S(1\times M) \arrow[u, "S\lambda_M"'] \arrow[d, "S(1\times f)"] \arrow[r, phantom, "\text{naturality of } S\lambda \text{ for } f"] & {} \\
 & & S1\times S1 \arrow[r, "t_{1,S1}"] \arrow[d, "\cdot"'] \arrow[dl, phantom, "f\text{ monoid homomorphism}"] & S(1\times S1) \arrow[r, "S\lambda_{S1}"] \arrow[d, phantom, "\text{Definition of “}\cdot\text{”}"] & SS1 \arrow[d, "\mu_1^S"] \\
M\times 1 \arrow[r, "\lambda_M"] \arrow[rrrr, rounded corners, to path={|- ([yshift=-4ex]\tikztotarget.south) \tikztonodes -- (\tikztotarget)}, "\alpha_1"'] & M \arrow[r, "f"] & S1 \arrow[rr, equal] & {} & S1
\end{tikzcd}
$$

The commutativity tells us that

$$\alpha_1 \cdot \mu_1^{M\times(-)} = \mu_1^S \cdot (\alpha * \alpha)_1.$$

Like in the case for the unit, we can extend the above diagram by the jointly epic

$$\big(M\times(M\times x) : M\times(M\times 1)\to M\times(M\times X)\big)_{x\in X}$$

and use the naturality of $\mu^{M\times(-)}, \mu^S$, α and $\alpha*\alpha$ to show that α preserves the monad multiplication. □

Lemma 5.46. *The forgetful functor $U : \mathsf{Mnd_f}(\mathcal{B}) \to \mathsf{Fun_f}(\mathcal{B})$ creates filtered colimits.*

Proof. Let $D : \mathcal{D} \to \mathsf{Mnd_f}(\mathcal{B})$, $Di = (M_i, \eta^i, \mu^i)$ be a filtered diagram. Take its colimit $M = \operatorname{colim} D$ with injections $\mathsf{in}_i : M_i \to M$ in $\mathsf{Fun_f}(\mathcal{B})$ and define a monad unit by

$$\eta \equiv \big(\mathrm{Id} \xrightarrow{\eta^i} M_i \xrightarrow{\mathsf{in}_i} M\big), \qquad \text{for any } i\in\mathcal{D}.$$

Similarly, define the monad multiplication $\mu : MM \to M$ as the unique natural transformation with

$$
\begin{tikzcd}
M_iM_i \arrow[r, "\mu^i"] \arrow[d, "\mathsf{in}_i * \mathsf{in}_i"'] & M_i \arrow[d, "\mathsf{in}_i"] \\
MM \arrow[r, dashed, "\mu"] & M
\end{tikzcd}
\qquad \text{for any } i\in\mathcal{D}.
$$

The filteredness of D proves the independence of the choice of i: for any other candidate $j \in \mathcal{D}$ choose an upper bound $m_{i,k} : M_i \to M_k \leftarrow M_j : m_{j,k}$ of M_i and M_j. Then we have a commutative diagram

$$
\begin{tikzcd}
 & M_i \arrow[d, "m_{i,k}"] \arrow[dr, "\mathsf{in}_i"] & \\
\mathrm{Id} \arrow[ur, "\eta^i"] \arrow[r, "\eta^k"] \arrow[dr, "\eta^j"'] & M_k \arrow[r, "\mathsf{in}_k"] & M \\
 & M_j \arrow[u, "m_{j,k}"'] \arrow[ur, "\mathsf{in}_j"'] &
\end{tikzcd}
$$

The left-hand triangles commute because $m_{i,k}, m_{j,k}$ are monad morphisms and the right-hand triangles because $m_{i,k}, m_{j,k}$ are connecting natural transformations of D and the in the colimit injections.

Note that $(M_iM_i)_{i\in\mathcal{D}}$ is a filtered diagram with colimit MM in $\mathsf{Fun_f}(\mathcal{B})$. Let us check the monad laws:

- Unit laws: the diagrams

$$\begin{tikzcd}
M_i \arrow[r, "\eta^i M_i"] \arrow[rr, bend left, equal] \arrow[dd, "\mathsf{in}_i"'] \arrow[dr, phantom, "\substack{\text{Naturality}\\ \text{of } \eta^i}"] & M_iM_i \arrow[r, "\mu^i"] \arrow[d, "M_i\mathsf{in}_i"] \arrow[dr, phantom, "\substack{\text{Definition}\\ \text{of } \mu}"] & M_i \arrow[dd, "\mathsf{in}_i"] \\
 & M_iM \arrow[d, "\mathsf{in}_i M"] & \\
M \arrow[ur, "\eta^i M"] \arrow[r, "\eta M"'] \arrow[ur, phantom, "\text{Def. } \eta"'] & MM \arrow[r, "\mu"'] & M
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M_i \arrow[r, "M_i\eta^i"] \arrow[rr, bend left, equal] \arrow[dd, "\mathsf{in}_i"'] \arrow[dr, "M_i\eta"'] \arrow[r, phantom, "\text{Def. } \eta"'] & M_iM_i \arrow[r, "\mu^i"] \arrow[d, "M_i\mathsf{in}_i"] \arrow[dr, phantom, "\substack{\text{Definition}\\ \text{of } \mu}"] & M_i \arrow[dd, "\mathsf{in}_i"] \\
 & M_iM \arrow[d, "\mathsf{in}_i M"] & \\
M \arrow[r, "M\eta"'] & MM \arrow[r, "\mu"'] & M
\end{tikzcd}$$

  commute. As the in_i are jointly epic, (M, η, μ) fulfills the unit laws.

- Associativity:

$$\begin{tikzcd}
M_iM_iM_i \arrow[rrr, "\mu^i M_i"] \arrow[dr, "\mathsf{in}_i * \mathsf{in}_i * \mathsf{in}_i"] \arrow[ddd, "M_i\mu^i"'] & & & M_iM_i \arrow[dl, "\mathsf{in}_i * \mathsf{in}_i"'] \arrow[ddd, "\mu^i"] \\
 & MMM \arrow[r, "\mu M"] \arrow[d, "M\mu"] & MM \arrow[d, "\mu"] & \\
 & MM \arrow[r, "\mu"] & M & \\
M_i \arrow[ur, "\mathsf{in}_i * \mathsf{in}_i"] \arrow[rrr, "\mu^i"'] & & & M_i \arrow[ul, "\mathsf{in}_i"']
\end{tikzcd}$$

  The outside commutes, and by definition of μ also all inner parts (except possibly for the middle square). As the $\mathsf{in}_i * \mathsf{in}_i * \mathsf{in}_i$ are jointly epic, the middle square commutes as well.

By definition of η and μ, each $\mathsf{in}_i : M_i \to M$ is a monad morphism. In fact, η and μ are the unique natural transformations making the diagrams (in the definition) commute, i.e. are the unique monad structure on M such that in_i is a monad morphism.

To see that (M, η, μ) is a colimiting cocone, consider another cocone $n_i : M_i \to N$ in $\mathsf{Mnd_f}(\mathcal{B})$. This induced a unique natural transformation $m : M \to N$ with $n_i = m \cdot \mathsf{in}_i$. To see that m is also a monad morphism, use the jointly epicness of the in_i:

$$m \cdot \eta = m \cdot \mathsf{in}_i \cdot \eta^i = n_i \cdot \eta^i = \eta^N,$$

Consider the following diagram:

$$\begin{tikzcd}
M_iM_i \arrow[rrr, "\mu^i"] \arrow[dr, "\mathsf{in}_i * \mathsf{in}_i"] \arrow[ddr, bend right, "n_i * n_i"'] & & & M_i \arrow[dl, "\mathsf{in}_i"'] \arrow[ddl, bend left, "n_i"] \\
& MM \arrow[r, "\mu"] \arrow[d, "m * m"'] \arrow[dr, phantom, "?"] & M \arrow[d, "m"] & \\
& NN \arrow[r, "\mu^N"] & N &
\end{tikzcd}$$

The outside commutes, because n_i is a monad morphism. The outer triangles commute on the level of $\mathsf{Fun_f}(\mathcal{B})$ and the upper part commutes because in_i is a monad morphism. Again, as the $\mathsf{in}_i * \mathsf{in}_i$ are jointly epic, the inner square commutes as well, hence m is a monad morphism. $\square$

Lemma 5.47. *The free monoid action defines a faithful functor $F : \mathsf{Mon} \hookrightarrow \mathsf{Mnd_f}(\mathsf{Set})$ by*

$$FM = T_M, \quad \big(F(f : M \to N)\big)_X = f \times \mathrm{id}_X : T_M X \to T_N X.$$

Proof. Obviously, $F(f : M \to N)$ is a natural transformation and F is a faithful functor $F : \mathsf{Mon} \to \mathsf{Fun_f}(\mathsf{Set})$. It remains to see that Ff is indeed a monad morphism $T_M \to T_N$. The diagram for the monad morphism property (5.3) commutes object-wise because $f : M \to N$ is a monoid homomorphism. $\square$

Lemma 5.48. *The inclusion functor $F : \mathsf{Mon} \hookrightarrow \mathsf{Mnd_f}(\mathsf{Set})$ is full.*

Proof. Any monad morphism $\varphi : T_M \to T_N$ has a monoid homomorphism in the component $\varphi_1 : M \to N$. The only thing left is to show that $\varphi = F\varphi_1$, i.e. $\varphi_X = \varphi_1 \times X$. For an arbitrary element $x : 1 \to X$ consider the naturality square of φ:

$$\begin{tikzcd}
M \times 1 \arrow[r, "\varphi_1"] \arrow[d, "M \times x"'] \arrow[dr, phantom, "\circlearrowleft"] & N \times 1 \arrow[d, "N \times x"] \\
M \times X \arrow[r, "\varphi_X"'] & N \times X
\end{tikzcd}$$

In other words, the commutativity says

$$\varphi_X(m, x) = (\varphi_1(m), x),$$

when considering φ_1 as a morphism $M \to N$. So $\varphi = F\varphi_1$ and F is full. $\square$

With this result, we can consider Mon as a full subcategory of $\mathsf{Mnd_f}(\mathsf{Set})$.

Lemma 5.49. *The inclusion functor $F : \mathsf{Mon} \hookrightarrow \mathsf{Mnd_f}(\mathsf{Set})$ preserves filtered colimits.*

Proof. Let $D : \mathcal{D} \to \mathsf{Mon}$ a filtered diagram. $U : \mathsf{Mnd_f}(\mathsf{Set}) \to \mathsf{Fun_f}(\mathsf{Set})$ creates filtered colimits by Lemma 5.46. I.e. filtered colimits are computed object-wise, just as in $\mathsf{Fun_f}(\mathsf{Set})$:

$$(\operatorname{colim} FD)X = \operatorname{colim}\big(i \mapsto (FDi)X\big) = \operatorname{colim}\big(i \mapsto T_{Di}X\big) = \operatorname{colim}\big(i \mapsto Di \times X\big)$$

So the remaining steps are

$$\operatorname{colim}\big(i \mapsto Di \times X\big) \stackrel{?}{=} \operatorname{colim}\big(i \mapsto Di\big) \times \operatorname{colim}\big(i \mapsto X\big) = (\operatorname{colim} D) \times X = \big(F(\operatorname{colim} D)\big)X,$$

which hold because filtered colimits commute with finite limits, products in particular. More detailed, filtered colimits can be characterized equivalently as those colimits which commute with finite limits [ARV10, Definition 2.1, Theorem 2.19]. $\square$

Proposition 5.50. *If M is not f.g. (respectively not f.p.) in Mon, then T_M is not f.g. (respectively not f.p.) in $\mathsf{Mnd_f}(\mathsf{Set})$.*

Proof. F preserves filtered colimits, and so also directed colimits [AR94, Theorem 1.5 and Corollary]. Monomorphisms in Mon are precisely the homomorphisms carried by monomorphisms. For a monoid monomorphism $f : M \to N$, each component $(Ff)_X = f \times \mathrm{id}_X$ of Ff is monic in Set, i.e. Ff is monic in $\mathsf{Mnd_f}(\mathsf{Set})$. Hence F preserves directed diagrams of monos.

In the following, we write $\mathsf{Mnd_f}$ as a shorthand for the category $\mathsf{Mnd_f}(\mathsf{Set})$. Consider the following cycle of equivalences for a monoid and an arbitrary diagram $D : \mathcal{D} \to \mathsf{Mon}$ which is a filtered or a directed diagram of monos:

$$
\begin{array}{ccc}
\mathsf{Mon}(M, \operatorname{colim} D) & \overset{\text{(iii)}}{\cong} & \operatorname{colim} \mathsf{Mon}(M, D(-)) \\
\text{(i)} \wr\| & & \\
\mathsf{Mnd_f}(FM, F \operatorname{colim} D) & & \|\wr\, \text{(ii)} \\
\text{(ii)} \wr\| & & \\
\mathsf{Mnd_f}(FM, \operatorname{colim} FD) & \overset{\text{(iv)}}{\cong} & \operatorname{colim} \mathsf{Mnd_f}(FM, FD(-))
\end{array}
$$

Some of the equivalences always hold:

(i) Because the inclusion functor is fully faithful.

(ii) Because F preserves both filtered colimits and directed colimits of monos.

Whereas some equivalences hold under certain circumstances:

(iii) holds iff M preserves the colimit of D.

(iv) holds iff $FM = T_M$ preserves the colimit of FD.

As the "vertical" equivalences (i) and (ii) always hold, we can conclude that (iii) iff (iv).

$$M \text{ not f.p.} \Rightarrow \exists D \text{ s.t. (iii) fails} \Rightarrow \text{(iv) fails for } FD \Rightarrow T_M = FM \text{ not f.p.}$$

Analogous, for M not finitely generated. □

Proposition 5.51. *If M is f.g. in Mon, then T_M is f.g. in $\mathsf{Mnd_f}(\mathsf{Set})$,*

Proof. Take a directed diagram of monos $D : \mathcal{D} \to \mathsf{Mnd_f}(\mathsf{Set})$, $C = \operatorname{colim} D$ and a monad morphism $f : T_M \to C$. We need to show that f factors through one of the Di, $i \in \mathcal{D}$. As D is in particular filtered, its colimit is computed like in $\mathsf{Fun_f}(\mathcal{B})$, by Lemma 5.46, i.e. object-wise on the level of Set. So

$$C1 = \operatorname{colim}(i \mapsto D_i 1)$$

in Set. As D is still filtered, the colimit is the same as in Mon, where the monoid structure of $D_i 1$ is as in Proposition 5.40. Furthermore, the diagram $(i \mapsto D_i 1) : \mathcal{D} \to \mathsf{Mon}$ is a directed diagram of monos, because monic monad morphisms have monic components, see Remark 5.12.

By Lemma 5.45, $f : T_M \to C$ corresponds to a monoid homomorphism $f_1 : M \to C1$. M is f.g., so it factors through some $D_i 1$:

$$
\begin{tikzcd}
M \arrow[r, "f_1"] \arrow[dr, dashed, "f_1^*"'] & C1 \\
& D_i 1 \arrow[u, "\mathsf{in}_{i,1}"']
\end{tikzcd}
$$

Applying Lemma 5.45 on $\mathsf{in}_{i,1} \cdot f_1^*$ and using that the isomorphism is natural for $\mathsf{in}_{i,1} : D_i 1 \to C1$ gives the desired factorization $f^* : M \to D_i$ with $\mathsf{in}_i \cdot f^* = f$ in $\mathsf{Mnd_f}(\mathsf{Set})$. $\square$

Corollary 5.52. *Let M be a f.g. but not f.p. monoid, e.g. that from [CRRT96, Example 4.5]. Then T_M is f.g. but not f.p. in $\mathsf{Mnd_f}(\mathsf{Set})$.*

So it is very likely, that the rational fixpoint and the locally finite fixpoint on $\mathsf{Mnd_f}(\mathsf{Set})$ and $H/\mathsf{Mnd_f}(\mathsf{Set})$ do not coincide.

5.4.2 What has been done already

Before we are able to apply the locally finite fixpoint to the scenario, we first need to recall all the required results and definitions from [AMV11a]

Assumption 5.53. Let $\mathcal{B}$ be an lfp category in which the coproduct injections are monic. Consider some finitary, mono-preserving $H : \mathcal{B} \to \mathcal{B}$.

Beside $\mathsf{Mnd_f}(\mathcal{B})$, we will also need the category of countably accessible monads on $\mathcal{B}$, denoted by $\mathsf{Mnd_c}(\mathcal{B})$, i.e. monads whose underlying functor preserves countably filtered colimits. In contrast to the finitary monads, $\mathsf{Mnd_c}(\mathcal{B})$ is not lfp, but still cocomplete and locally $\aleph_1$-presentable.[1]

A central role plays the free monad of a finitary endofunctor.

Definition 5.54 (Free monad)**.** For a finitary endofunctor H, free H-algebras

$$\varphi_X : HF^H X \to F^H \tag{5.16}$$

exist for all objects X of $\mathcal{B}$, by [Adá74]. F^H is the monad on $\mathcal{B}$ of free H-algebras and is also a free monad on H, by [Bar70], and F^H is a finitary monad. Its unit is denoted by

$$\hat{\eta} : \mathrm{Id} \to F^H$$

and the family φ from (5.16) form a natural transformation

$$\varphi : HF^H \to F^H.$$

Their composition is the universal arrow

$$\hat{\kappa} \equiv \left(H \xrightarrow{H\hat{\kappa}} HF^H \xrightarrow{\varphi} F^H\right)$$

with the following universal property: for each monad S and natural transformation $f : H \to S$, there exists a unique monad morphism $\bar{f} : F^H \to S$ such that

$$\begin{tikzcd} H \arrow[r, "\hat{\kappa}"] \arrow[dr, "f"'] & F^H \arrow[d, dashed, "\bar{f}"] \\ & S \end{tikzcd}$$

From [Adá74], we also have

$$F^H = HF^H + \mathrm{Id} \text{ with coproduct injections } \varphi \text{ and } \hat{\eta}. \tag{5.17}$$

[1] The notion of λ-presentable objects and locally λ-presentable categories, for λ a regular cardinal, is not relevant for this work. We only need to consider the case $\lambda = \aleph_0$, to obtain finitely presentability. For the general case, see [AR94, Section 1.B].

The previous definition also tells us, that $F^{(-)}$ is left adjoint to the forgetful functor $\mathsf{Mnd_f}(\mathcal{B}) \to \mathsf{Fun_f}(\mathcal{B})$. Moreover, $\mathsf{Mnd_f}(\mathcal{B})$ is monadic over $\mathsf{Fun_f}(\mathcal{B})$, as one readily shows using Beck's theorem. As $F^{(-)} : \mathsf{Fun_f}(\mathcal{B}) \to \mathsf{Mnd_f}(\mathcal{B})$ preserves any colimits, we have that the monad $UF^{(-)}$ on $\mathsf{Fun_f}(\mathcal{B})$ is finitary, and by [AR94, 2.78 Theorem and Remark] we obtain:

Corollary 5.55. $\mathsf{Mnd_f}(\mathcal{B})$ *is lfp and in particular cocomplete.*

The endofunctor, for which we will consider the locally finite fixpoint later, works on the category

$$H/\mathsf{Mnd_f}(\mathcal{B}),$$

in which the objects are H-pointed finitary monads, i.e. finitary monads (M, μ^M, η^M) together with a natural transformation $H \to M$, the *pointing*. By the universal property of the free monad, we have

$$H/\mathsf{Mnd_f}(\mathcal{B}) \cong F^H/\mathsf{Mnd_f}(\mathcal{B}),$$

i.e. we work on a coslice category. As free constructions preserve colimits, we have

$$F^{H+V} \cong F^H \oplus F^V,$$

where $\oplus$ denotes the coproduct in $\mathsf{Mnd_f}(\mathcal{B})$. Combining the construction of the free monad and that of the coslice category gives us that $(F^{H+V}, H : \hat{\kappa} \cdot \mathsf{inl})$ together with the unit $\hat{\kappa} \cdot \mathsf{inr} : V \to F^{H+V}$ is the free H-pointed monad for finitary $\mathcal{B}$-endofunctor V.

For every H-pointed monad (B, β), one gets a natural transformation

$$b^+ = [\mu^B \cdot \beta B, \eta^B] : HB + \mathrm{Id} \to B.$$

Lemma 5.56 ([GLM05]). *For every H-pointed monad (B, β), the $\mathcal{B}$-endofunctor $HB + \mathrm{Id}$ carries a canonical monad structure with the unit* $\mathsf{inr} : \mathrm{Id} \to HB + \mathrm{Id}$ *and the multiplication*

$$\underbrace{HB(HB + \mathrm{Id}) + HB + \mathrm{Id}}_{=(HB+\mathrm{Id})(HB+\mathrm{Id})} \xrightarrow{HBb^+ + HB + \mathrm{Id}} HBB + HB + \mathrm{Id} \xrightarrow{[H\mu^B, HB] + \mathrm{Id}} HB + \mathrm{Id}.$$

Definition 5.57. The monad $HB + \mathrm{Id} : \mathcal{B} \to \mathcal{B}$ has the H-pointing

$$\mathsf{inl} \cdot H\eta^B : H \to HB + \mathrm{Id}.$$

In total, we have an endofunctor $\mathscr{H} : H/\mathsf{Mnd_c}(\mathcal{B}) \to H/\mathsf{Mnd_c}(\mathcal{B})$ which operates on objects by

$$\mathscr{H}(B, \beta) = (HB + \mathrm{Id}, \mathsf{inl} \cdot H\eta^B),$$

see also [GLM05] and [MM06, Lemma 5.2]. This functor also restricts to H-pointed finitary monads:

$$\mathscr{H}_f : H/\mathsf{Mnd_f}(\mathcal{B}) \to H/\mathsf{Mnd_f}(\mathcal{B}).$$

Proposition 5.58. *The functor $\mathscr{H}_\mathsf{f}$ is finitary.*

Proof. Filtered colimits in $H/\mathsf{Mnd_f}(\mathcal{B}) \cong F^H/\mathsf{Mnd_f}(\mathcal{B})$ are as in $\mathsf{Mnd_f}(\mathcal{B})$, because the coslice category is equivalent to the Eilenberg-Moore category for the finitary $\mathsf{Mnd_f}(\mathcal{B})$-monad $F^H \oplus (-)$.

By Lemma 5.46, filtered colimits in $\mathsf{Mnd_f}(\mathcal{B})$ are as in $\mathsf{Fun_f}(\mathcal{B})$, and there object-wise. And the functor $B \mapsto HB + \mathrm{Id}$ is finitary on $\mathsf{Fun_f}(\mathcal{B})$ since H is finitary on $\mathcal{B}$. $\square$

Note, that for every object X the finitary $\mathcal{B}$-endofunctor $H(-) + X$, has a terminal coalgebra

$$TX \xrightarrow{\cong} HTX + X, \quad [\tau_X^T, \eta_X^T] : HTX + X \to TX.$$

By [AAMV03], T is a monad with unit η^T and is called the *free completely iterative* monad. It also is H-pointed via

$$H \xrightarrow{H\eta^T} HT \xrightarrow{\tau^T} T$$

This monad is only countably accessible but not finitary. So it lives in $\mathsf{Mnd_c}(\mathcal{B})$, which is the only reason why we consider $\mathsf{Mnd_c}(\mathcal{B})$ at all in this work.

Proposition 5.59 ([MM06, Theorem 5.4])**.** *T is the final coalgebra for $\mathcal{H}$.*

Most of the time, we are considering $\mathcal{H}$-coalgebras carried by some F^{H+V} with the pointing

$$H \xrightarrow{\mathsf{inl}} H + V \xrightarrow{\hat{\kappa}} F^{H+V}.$$

Then, F^{H+V} is mapped to $\mathcal{H}F^{H+V} = HF^{H+V} + \mathrm{Id}$ equipped with the pointing

$$\psi \equiv \big(H \xrightarrow{H\hat{\eta}} HF^{H+V} \xrightarrow{\mathsf{inl}} HF^{H+V} + \mathrm{Id}\big)$$

These $\mathcal{H}$-coalgebras with a free carrier are the coalgebras of interest, because they encode recursive program schemes.

Definition 5.60. A *recursive program scheme* (rps) of type H is a natural transformation

$$e : V \to F^{H+V}$$

where V is a finitely presentable object of $\mathsf{Fun_f}(\mathcal{B})$. An rps is called *guarded* if it factorizes through the summand $HF^{H+V} + \mathrm{Id}$ of the corpoduct in (5.17):

$$F^{H+V} = (H + V)F^{H+V} + \mathrm{Id} = HF^{H+V} + VF^{H+V} + \mathrm{Id}.$$

In other words, for each such e we have a e_0 with

$$\begin{tikzcd} V \arrow[r, "e"] \arrow[dr, dashed, "e_0"'] & F^{H+V} \\ & HF^{H+V} + \mathrm{Id}\mathcal{H}F^{H+V} \arrow[u, "{[\varphi \cdot \mathsf{inl}, \hat{\eta}]}"'] \end{tikzcd}$$

By Assumption 5.53, coproduct injections are monic, and e_0 is necessarily unique, i.e. e and e_0 are in bijective correspondence.

Observation 5.61. In total we have the following chain of bijective correspondences:

$$\frac{\dfrac{e : V \to F^{H+V} \text{ guarded rps}}{e_0 : V \to HF^{H+V} + \mathrm{Id} \text{ natural transformation}}}{\bar{e}_0 : F^{H+V} \to \mathcal{H}F^{H+V} \text{ morphism in } H/\mathsf{Mnd_f}(\mathcal{B})}$$

Important properties are furthermore the following:

- If V is f.p. in $\mathsf{Fun_f}(\mathcal{B})$, then F^{H+V} is f.p. in $H/\mathsf{Mnd_f}(\mathcal{B})$.
- $\mathsf{Mnd_f}(\mathcal{B})$ is closed in $\mathsf{Mnd_c}(\mathcal{B})$ under subobjects and strong quotients.

We will now consider different diagrams in $\mathsf{Coalg}\mathcal{H}$ in order to obtain different monads with different properties.

Definition 5.62. The following diagrams are of interest:

- Let EQ be the coalgebras $F^{H+V} \to \mathcal{H}F^{H+V}$, V f.p. in $\mathsf{Fun_f}(\mathcal{B})$.
- By EQ_1, denote the closure of EQ under coequalizers in $\mathsf{Coalg}\mathcal{H}$ (note that $\mathsf{Coalg}\mathcal{H}$ is cocomplete).
- The closure of EQ under strong quotients is defined as EQ_2.

In [AMV11a], they consider the colimit of EQ_1,

$$(S^H, s : S^H \to \mathcal{H}S^H) := \operatorname{colim} \mathsf{EQ}_1$$

and call it the *second order rational monad*, which is finitary. Its image in the final coalgebra T, is called the *context-free monad* C^H, and is constructed by factorizing the unique coalgebra homomorphism $s^* : S^H \to T$ according to Proposition 4.20.

Furthermore, it is proven that C^H is precisely the colimit of EQ_2. As finitary monads are closed under epimorphisms, C^H is finitary as well, so we get the following picture:

$$\begin{tikzcd}
S^H \arrow[rr, "s^*"] \arrow[dr, two heads] & & T = \nu\mathcal{H} \\
& C^H \arrow[ur, tail] &
\end{tikzcd}$$

5.4.3 How the LFF helps

First of all, let us see that everything from Assumption 4.21 is met:

- The category $H/\mathsf{Mnd_f}(\mathcal{B})$ is lfp.
- The endofunctor $\mathcal{H}_\mathfrak{f}$ is finitary by Proposition 5.58.
- $\mathcal{H}_\mathfrak{f}$ preserves monomorphisms [AMV10, Corollary 2.20].

S^H looks like the rational fixpoint in disguise and with the relation between the rational fixpoint and the LFF in mind, C^H remembers us of the LFF of $\mathcal{H}_\mathfrak{f}$. To actually prove that, we just need to show that EQ_2 consists of precisely those $\mathcal{H}_\mathfrak{f}$-coalgebras with finitely generated carrier.

However, we will need to use the properties we know about $\mathsf{Mnd_f}(\mathsf{Set})$, extensively, so we will restrict to the case $\mathcal{B} = \mathsf{Set}$ from now on.

Remark 5.63. Everything from Assumption 5.53 is met. Furthermore note the following properties of Set:

- The f.p. V in $\mathsf{Fun_f}(\mathsf{Set})$ are the quotients of polynomial functors.
- The polynomial functors are epi projectives (recall Definition 4.47).
- Any f.g. M of $H/\mathsf{Mnd_f}(\mathcal{B})$ is the quotient of a F^{H+V} for V f.p. in $\mathsf{Fun_f}(\mathcal{B})$

Lemma 5.64. *The strong epis in* $\mathsf{Mnd_f}(\mathsf{Set})$ *have surjective components.*

Proof. Let $q : M \to N$ be a strong epi in $\mathsf{Mnd_f}(\mathsf{Set})$. Consider the (strong epi,mono)-factorizations of the components in $\mathsf{Fun_f}$:

$$\begin{tikzcd}
M \arrow[r, two heads, "e"] \arrow[rr, bend left, "q"] & I \arrow[r, tail, "m"] & N
\end{tikzcd}$$

The factorization lifts further to $\mathsf{Mnd_f}(\mathsf{Set})$, i.e. we have factorized the monad morphism q into an epi e and a mono m in $\mathsf{Mnd_f}(\mathsf{Set})$. As any strong epi is also extremal, we get that m is an isomorphism. Hence q has epic components. □

Lemma 5.65. *$\mathscr{H}_{\mathsf{f}}$ maps strong epimorphisms to morphisms carried by a strong epi natural transformation.*

Proof. After Proposition 3.9, it remains to show that strong epis from $\mathsf{Mnd_f}(\mathsf{Set})$ are preserved. Consider the strong epi $q : A \to B$ in $\mathsf{Mnd_f}(\mathsf{Set})$. By Lemma 5.64, the components q_X are epic. All Set-endofunctors preserve epis, so $Hq_X + \mathrm{Id}$ is epic for any set X and so the natural transformation $Hq + \mathrm{Id}$ is epic as well. □

Lemma 5.66. *Any $\mathscr{H}_{\mathsf{f}}$-coalgebra $((B, \beta : H \to B), b : (B, \beta) \to \mathscr{H}(B, \beta))$ with (B, β) f.g. is the strong quotient of a coalgebra from* EQ.

The rough proof idea is similar to that of Proposition 4.50.

Proof. By Remark 5.63, (B, β) is the strong quotient of a $(F^{H+V}, \hat{\kappa} \cdot \mathsf{inl})$, which again is a quotient of $(F^{H+P}, \hat{\kappa} \cdot \mathsf{inl})$, where P a polynomial functor and therefore an epi-projective in $\mathsf{Fun_f}(\mathcal{B})$.

$$\begin{tikzcd}
(F^{H+P}, \hat{\kappa} \cdot \mathsf{inl}) \arrow[r, two heads, "q_P"] \arrow[rr, two heads, bend right, "=: q"'] & (F^{H+V}, \hat{\kappa} \cdot \mathsf{inl}) \arrow[r, two heads, "q_V"] & (B, \beta) \arrow[r, "b"] & \mathscr{H}(B, \beta)
\end{tikzcd}$$

This corresponds to a natural transformation $\overline{b \cdot q} : P \to HB + \mathrm{Id}$. As P is projective and by Lemma 5.65 $\mathscr{H}_{\mathsf{f}}q$ is epic as a natural transformation, we get a natural transformation $p : P \to HF^{H+P} + \mathrm{Id}$.

$$\begin{tikzcd}
P \arrow[r, dashed, "p"] \arrow[dr, "\overline{b \cdot q}"'] & HF^{H+P} + \mathrm{Id} \arrow[d, two heads, "Hq + \mathrm{Id}"] \\
& HB + \mathrm{Id}
\end{tikzcd}
\iff
\begin{tikzcd}
(F^{H+P}, \hat{\kappa} \cdot \mathsf{inl}) \arrow[r, "\bar{p}"] \arrow[d, two heads, "q"'] & \mathscr{H}_{\mathsf{f}}(F^{H+P}, \hat{\kappa} \cdot \mathsf{inl}) \arrow[d, "Hq + \mathrm{Id}"] \\
(B, \beta) \arrow[r, "b"] & \mathscr{H}_{\mathsf{f}}(B, \beta)
\end{tikzcd}$$

So the coalgebra b is the strong quotient of $\bar{p}$, which is in EQ. □

This result implies that EQ_2 – the closure of EQ under strong quotients – consists of precisely the $\mathscr{H}_{\mathsf{f}}$-coalgebras with finitely generated carrier.

Corollary 5.67. *The context-free monad C^H is the locally finite fixpoint of $\mathscr{H}_{\mathsf{f}}$.*

Note that the free completely iterative monad T is accessible but not finitary, i.e. it cannot be the final coalgebra for $\mathscr{H}_{\mathsf{f}}$. But on the other hand, we know that $\nu\mathscr{H}_{\mathsf{f}}$ must exists, because $\mathscr{H}_{\mathsf{f}}$ is finitary and $H/\mathsf{Mnd_f}(\mathsf{Set})$ is lfp. It is an open, how a characterization of $\nu\mathscr{H}_{\mathsf{f}}$ could look like.

But luckily, we do not need to care about that because C^H is a submonad of both $\nu\mathscr{H}_{\mathsf{f}}$ and T and because C^H is the image of S^H in $\nu\mathscr{H}_{\mathsf{f}}$ and in T. In a picture:

$$\begin{tikzcd}
S^H \arrow[r, "s^{*\prime}"] \arrow[rr, bend left, "s^*"] \arrow[dr, two heads] & \nu\mathscr{H}_{\mathsf{f}} \arrow[r] & T \\
& C^H \arrow[u, tail] \arrow[ur, tail, "c"'] &
\end{tikzcd}$$

The finality of $\nu\mathscr{H}_{\mathsf{f}}$ in $H/\mathsf{Mnd_f}(\mathsf{Set})$ and of T in $H/\mathsf{Mnd_c}(\mathsf{Set})$ makes all parts commute.

6 Conclusion

6.1 Main Results

We now have an abstract and uniform description of the collection of finite behaviours of H-coalgebras, namely the *locally finite fixpoint* of H: beside being a fixpoint of H, we get a subcoalgebra of the final coalgebra that contains precisely the behaviours of H-coalgebras with a finitely generated carrier. We have seen this result both on an abstract level by the universal property of the final lfg coalgebras and concretely for algebraic categories.

The relation to existing work is clear: the LFF is the image of the rational fixpoint. In the many examples of the rational fixpoint, in which the finitely generated objects are finitely presentable, the two notions coincide. Those examples include the category of sets, of vector spaces, of nominal sets, and of finitary set-endofunctors.

6.2 Future Work

However, there are still some interesting applications of f.g. carried coalgebras, that are left open for future work.

In [GMS14], a tape automaton – which is very similar to a Turing-machine – is defined coalgebraically and as an instance of the generalized powerset construction. It still needs to be investigated how this fits in the framework of the LFF in order to see how one can express their behaviours – the class of semi-decidable languages – as the locally finite fixpoint.

The generalized determinization itself considers monads over Set. It is open, how the results can be generalized to monads over lfp categories. This question is interesting, because in our other main application – that of algebraic trees – we have seen a similar phenomenon: guarded recursive program schemes are precisely natural transformations

$$e_0 : V \to HF^{H+V} + \mathrm{Id}.$$

These were lifted to a pointed monad morphism using the universal property of $F^{H+(-)}$

$$\bar{e}_0 : F^{H+V} \to HF^{H+V} + \mathrm{Id},$$

which looks precisely like the generalized determinization for the transition type $\mathcal{H} = H\cdot(-)+\mathrm{Id}$ and the computational type $F^{H+(-)}$.

Note that in all considered examples of Set^T or $\mathsf{Mnd_f}(\mathsf{Set})$, though we know about finitely generated objects that are not finitely presented, we still have no certainty that the rational fixpoint is not a subcoalgebra of the final coalgebra and that the rational and the LFF differ. But nevertheless, we were able to prove in this thesis that the LFF is definitely the appropriate notion when talking about finitely generated coalgebraic systems.

List of Symbols

Categories

Special morphisms

Other notions

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