Finitary Corecursion for the Infinitary λ-Calculus

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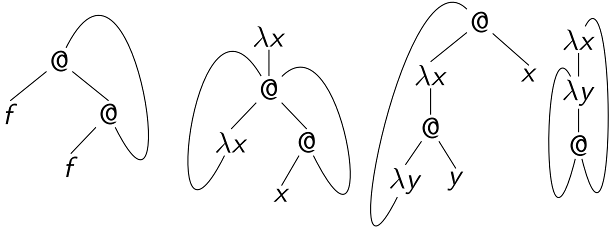
Motivation

Infinite λ-trees having a finite description



Questions

- Corecursion- & Coinduction principle?
- Closed under corecursive functions? E.g. substitution?
- Described by a universal property?

$\Rightarrow$ Rational Fixpoint of a Nom-Functor

History & Table of contents

λ-calculus (1930s)

FM-sets (1920s)

λ-terms modulo α-equivalence (1999)

Coalgebras

Infinite λ-trees modulo α-equivalence (2013)

Rational fixed point (2006)

Rational λ-trees modulo α-equivalence (this talk, 2015)

Nominal sets

$$\mathcal{V} = \{v_0, v_1, v_2, \ldots\} = \text{ some infinite countable set of variables}$$
$$\mathfrak{S}(\mathcal{V}) = \text{ set of finite permutations of } \mathcal{V}$$

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- X some set
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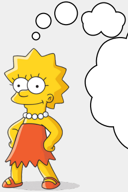
terms with finitely many free variables

Definition: $S \subseteq \mathcal{V}$ supports $x \in X$

For all $\pi \in \mathfrak{S}(\mathcal{V})$ with $\pi(v) = v\ \forall v \in S$, it is $\pi \cdot x = x$.



Leave S fix
$\Downarrow$
Leave x fix

Category of Nominal sets

Category Nom

- Objects: nominal sets
- Morphisms: equivariant maps $f : (X, \cdot) \to (Y, \star)$,
  i.e. maps $f : X \to Y$ with

$$f(\pi \cdot x) = \pi \star f(x).$$

Abstraction

v fresh for $x \Leftrightarrow v \notin \mathrm{supp}(x)$

Definition: Abstraction-Functor

$$(v_1, x_1) \sim_\alpha (v_2, x_2)$$
$$\Leftrightarrow$$
$$(v_1\ z)\ x_1 = (v_2\ z)\ x_2 \text{ with } z \text{ fresh for } v_1, v_2, x_1, x_2$$

- Nominal set $[V]X = (V \times X)/{\sim_\alpha}$.
- $\langle v \rangle x = {\sim_\alpha}$-class of (v, x).

$\Rightarrow$ v is fresh for $\langle v \rangle x$.

Gabbay and Pitts 1999

Algebraic description

Nom-Endofunctor

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X$$

$\Rightarrow$ Initial L-algebra $=$ λ-terms

$$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

$\Rightarrow$ Initial L_α-algebra $=$ λ-terms modulo α-equivalence.

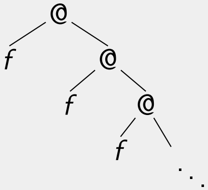
Gabbay and Pitts 1999

Where do infinite λ-trees come from?

Consider the fixpoint of (possibly) infinite β-reductions:

$$Y\ f \to_\beta^* f(Y\ f) \to_\beta^* f(f(Y\ f)) \to_\beta^* f(f(f(Y\ f)))$$

$$\to_\beta^* \ldots \to_\beta^*$$



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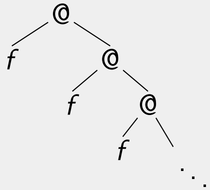
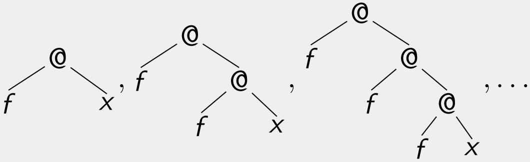


$\Rightarrow$ Can be described as a Cauchy-sequence of finite trees:



α-equivalence = component-wise on Cauchy-sequences.

Coalgebraic description

Final L_α-coalgebra

- infinite λ-trees
- modulo α-equivalence
- only finitely many free variables
- possibly infinitely many bound variables

Kurz et al. 2013

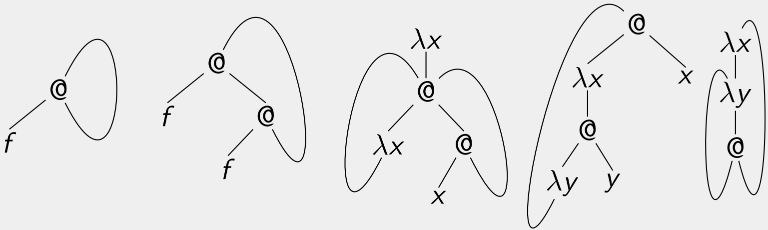
Rational λ-tree

Definition: λ-tree rational

If only finitely many subtrees (up to isomorphism).

Definition: λ-tree rational modulo α-equivalence

One element in the α-equivalence class is rational.



Rational fixpoint in general

$\mathcal{C}$ locally finitely presentable category
$F : \mathcal{C} \to \mathcal{C}$ finitary
$\mathrm{CoAlg} F$ = category of F-coalgebras
$\mathrm{CoAlg}_f F$ = coalgebras with finitely presentable carrier

Theorem (Adámek et al. 2006)

$$(\varrho F, r) := \mathrm{colim}(\mathrm{CoAlg}_f F \hookrightarrow \mathrm{CoAlg} F)$$

- $r : \varrho F \to F \varrho F$ is an isomorphism
- $(\varrho F, r)$ is the final locally finitely presentable F-Coalgebra

Examples

Regular Languages, Rational Streams, Rational Formal Power Series, Rational Σ-Trees, ...

Rational fixpoint in Nom

- Nom is locally finitely presentable
- finitely presentable = orbit finite
- locally finitely presentable F-coalgebra = locally orbit-finite

Proposition

$$\varrho F = \bigcup_{(A,a) \in \mathsf{CoAlg}_f F} a^\dagger[A] \quad \subseteq \nu F$$

with $a^\dagger : (A, a) \to (\nu F, t)$ unique coalgebra homomorphism.

Rational λ-trees coalgebraically

$$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X \qquad LX = \mathcal{V} + \mathcal{V} \times X + X \times X$$

Theorem

$\varrho L_\alpha =$ rational λ-trees modulo α-equivalence

Proof (Sketch).

- $\supseteq$ Construct orbit-finite coalgebra for rational λ-tree.
- $\subseteq$ Given orbit-finite (X, a) and root $\in X$.

$$m := \max_{x \in X} |\operatorname{supp}(x)|.$$

Take W s.t. $\operatorname{supp}(\text{root}) \subseteq W \subseteq \mathcal{V}, |W| = m + 1$.
Build rational λ-tree via finite L-coalgebra in Set with carrier
$C = \{x \in X \mid \operatorname{supp}(x) \subseteq W\}$ □

Finitely many subtrees, but how many?

Theorem (bound for $|C|$)

$m := \max_{x \in X} |\text{supp}(x)|$, $n =$ number of orbits of X.
Number of subtrees of $a^\dagger(\mathit{root})$ is bounded by

$$n \cdot (m+1)!$$

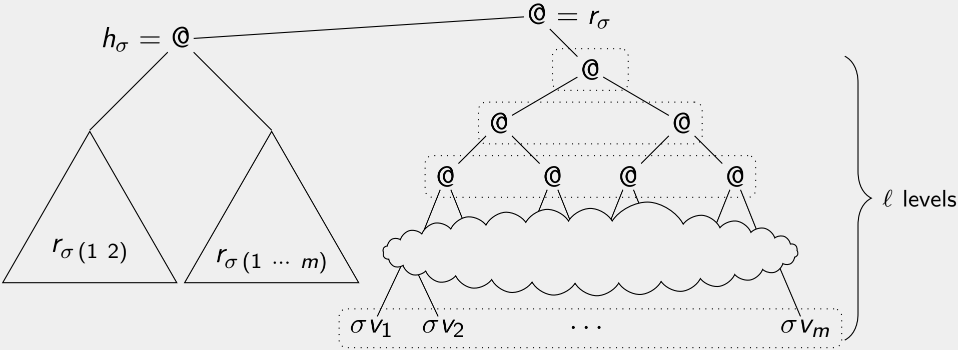
Example

Family of examples indexed by ℓ

m free variables,
$\ell + 2$ orbits.

Number of subtrees

$$2 \cdot m! + \sum_{i=1}^{\ell} \frac{m!}{(m - 2^{i-1})!}$$



Corecursion on rational λ-trees

Example: Substitution on ϱL_α

Wanted: $\mathsf{subs}_{\mathsf{rat}} : \varrho L_\alpha \times \mathcal{V} \times \varrho L_\alpha \to \varrho L_\alpha$

Needed: locally orbit-finite L_α-coalgebra on $\varrho L_\alpha \times \mathcal{V} \times \varrho L_\alpha$

Trick: Parameterize by orbit-finite coalgebras

$$A \xrightarrow{a} L_\alpha A, \qquad B \xrightarrow{b} L_\alpha B$$

Define L_α-coalgebra on $B + A \times \mathcal{V} \times B$:

$$\begin{array}{ccc}
B + A \times \mathcal{V} \times B & \longrightarrow & L_\alpha(B + A \times \mathcal{V} \times B) \\
{\scriptstyle [b^\dagger, \mathsf{subs}_{A,B}]} \downarrow & & \downarrow {\scriptstyle L_\alpha([b^\dagger, \mathsf{subs}_{A,B}])} \\
\varrho L_\alpha & \xrightarrow{\quad r \quad} & L_\alpha(\varrho L_\alpha)
\end{array}$$

Extend $\mathsf{subs}_{A,B}$ to $\mathsf{subs}_{\mathsf{rat}}$ using union.

Conclusion & Future Work

Conclusion

- Rational λ-trees (modulo α) as the rational fixpoint of L_α
- "Tight" bound on finite coalgebra constructed from orbit-finite coalgebra
- Corecursive definitions on rational λ-trees, e.g. substitution

Future Work

- Generalization to other finitary Nom-Endofunctors
- Abstract GSOS-rules in Nom
- Solutions of Higher-order recursion schemes in Nom
- Connection to presheaf-based approach to infinitary & rational λ-trees

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Murdoch Gabbay and Andrew Pitts. A new approach to abstract syntax involving binders. In Giuseppe Longo, editor, *Proceedings of the Fourteenth Annual IEEE Symp. on Logic in Computer Science, LICS 1999*, pages 214–224. IEEE Computer Society Press, July 1999.

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