Bisimilar States in Uncertain Structures

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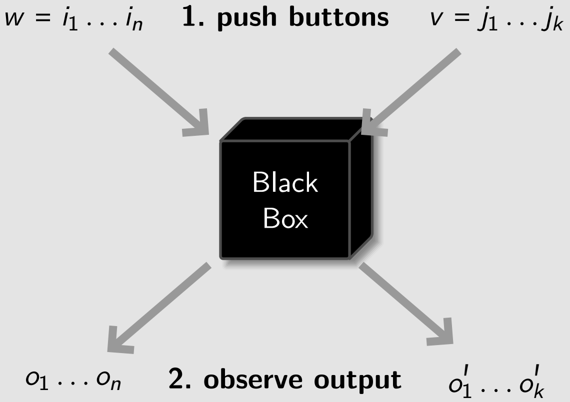
Context: Automata Learning

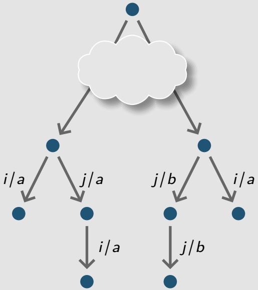


Algorithmic Challenge

Do the input sequences w and v bring the black box into the same internal state? Angluin '87

$L^{\#}$ Algorithm: Observation Tree

Combine all I/O-sequences in a single tree:



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Which states in the tree belong to the same state in the black box?

Vaandrager, Garhewal, Rot, Wißmann '22

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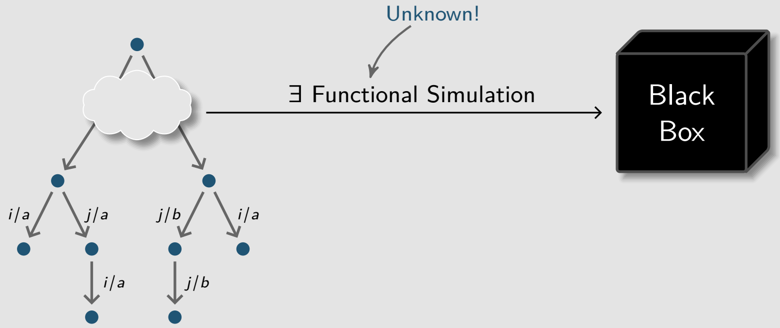

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Which states in the tree belong to the same state in the black box?

$\Longleftrightarrow$ Which states are identified by the functional simulation?

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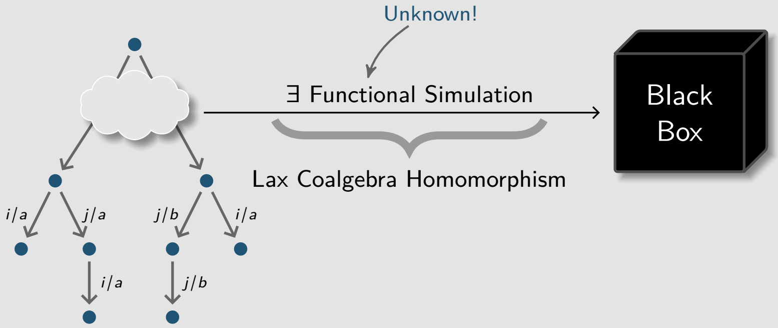

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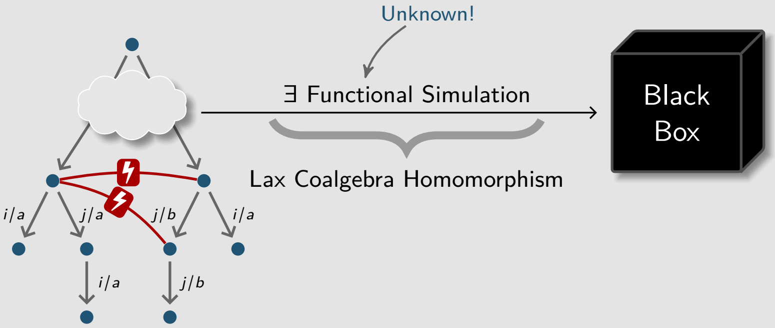

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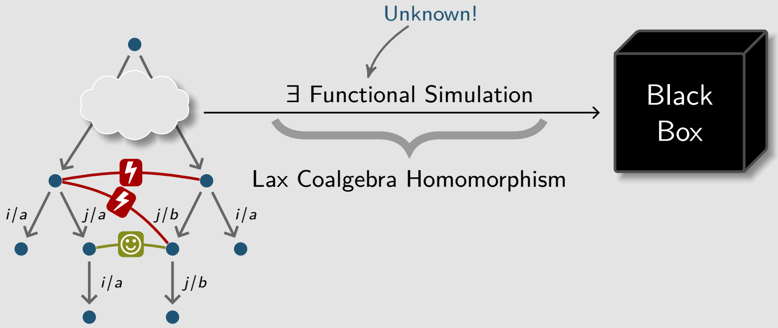

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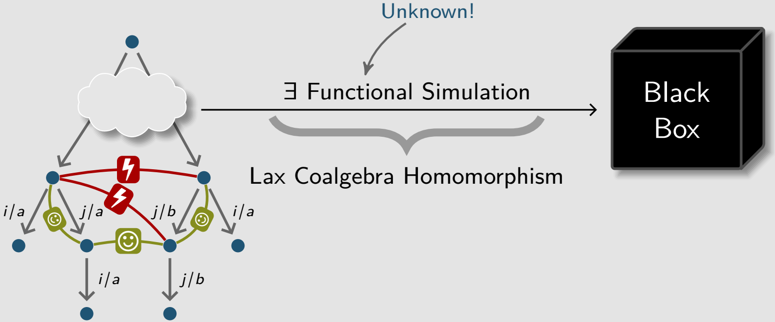

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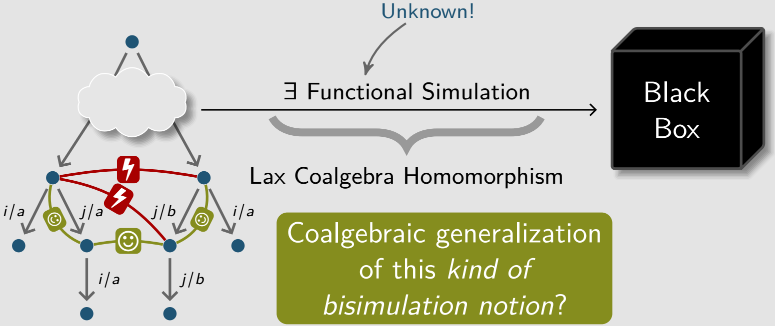

Coalgebraic generalization of this *kind of bisimulation notion*?

Algorithmic Challenge

Which states in the tree belong to the same state in the black box?

⟺ Which states are identified by the functional simulation?

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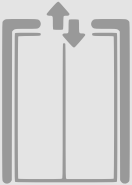
Contributions

- Definition of *uncertain bisimulation* on F-coalgebras
  … through the lense of relation liftings
- Instances:
  - Partial Mealy Machines (as in $L^{\#}$)
  - Suspension Automata (as in ioco conformance testing)

Interesting Properties

- ✓ reflexivity ✓ symmetry ~~transitivity~~
- Half a characterization via lax coalgebra morphisms
- Characterization via coalgebraic simulations

Relation Liftings



Lifting of a functor $F\colon \mathsf{Set} \to \mathsf{Set}$

sends relations on X to relations on FX

Standard Relation Lifting of $F\colon \mathsf{Set} \to \mathsf{Set}$

For $R \subseteq X \times X$, define $\hat{F}R \subseteq FX \times FX$ by:

$$\hat{F}R := \{(F\pi_1(t), F\pi_2(t)) \mid t \in FR\}$$

for projection maps $\pi_1, \pi_2\colon R \to X$

Coalgebraic Bisimulation

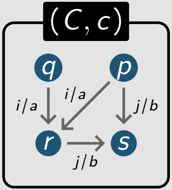
Bisimulation on $c\colon C \to FC \quad \equiv \quad r\colon R \to \hat{F}R$ with $p(r) = c$

Relation $R \subseteq C \times C$ such that

$$(x, y) \in R \quad \Longrightarrow \quad (c(x), c(y)) \in \hat{F}R \subseteq FX \times FX$$

Example: Partial Mealy Machines

$\mathcal{M}X := (I \rightharpoonup O \times X)$ (partial maps)
for fixed inputs I and outputs O:
Coalgebraic Bisimulation = Bisimulation on LTSs



Varying the lifting:

Lifting Rel → Rel	Coalgebras for the lifting
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$R \mapsto {\sqsubseteq} \circ \hat{F}R \circ {\sqsubseteq}$	Simulation	Hughes, Jacobs '04

Order on the functor '$\sqsubseteq$' $\subseteq FX \times FX$

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Lifting Rel → Rel	Coalgebras for the lifting	
$R \mapsto \hat{F}R$	Bisimulation	
$R \mapsto \sqsubseteq \circ \hat{F}R \circ \sqsubseteq$	Simulation	Hughes, Jacobs '04
$R \mapsto \sqsubseteq \circ \hat{F}R \circ \sqsupseteq$	Uncertain Bisimulation	*This Paper!*

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Equipping Functors with Partial Order

Assumption: fix a functor $F_{\mathsf{Pos}}\colon \mathsf{Set} \to \mathsf{Pos}$ ... or explicitly:

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Example: Partial Mealy Machines with inputs I and outputs O

- $\mathcal{M}X := (I \rightharpoonup O \times X)$ (partial maps)
- $f_1 \sqsubseteq_{\mathcal{M}X} f_2 \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall i \in \mathrm{dom}(f_1)\colon\ i \in \mathrm{dom}(f_2)\ \&\ f_1(i) = f_2(i)$

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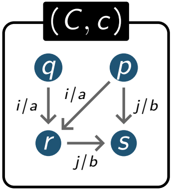
Example: $\mathcal{M}$-coalgebra $c\colon C \to \mathcal{M}C$, $I = \{i, j\}$, $O = \{a, b\}$



$c(q) \sqsubseteq c(p)$

$c(q) \not\sqsubseteq c(r)$
$c(q) \not\sqsupseteq c(r)$

Uncertain Bisimulation

... is a coalgebra for $\sqsubseteq \circ \hat{F}(-) \circ \sqsupseteq \colon \mathsf{Rel} \to \mathsf{Rel}$:

Lemma. An uncertain bisimulation on $c\colon C \to FC$ is a relation $R \subseteq C \times C$ such that

$$\forall (x, y) \in R \quad \exists t \in FR\colon \quad c(x) \sqsubseteq F\pi_1(t) \text{ and } c(y) \sqsubseteq F\pi_2(t)$$

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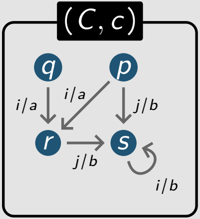
Properties: ✔ reflexivity ✔ symmetry ~~transitivity~~

Example $F := \mathcal{M}$. A reflexive $R \subseteq C \times C$ is an uncertain bisimulation iff

$$(x,y) \in R \;\&\; x \xrightarrow{i/o} x' \;\&\; y \xrightarrow{i/o'} y'$$
$$\Downarrow$$
$$o = o' \;\&\; (x', y') \in R$$



$R = \{(q,p)\} \cup \mathsf{IdRel}_C$

Instance: IOCO Conformance Testing

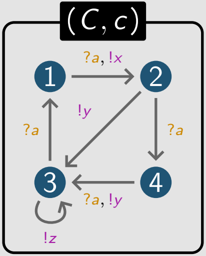
Suspension Automata = Coalgebras for

$$\mathcal{S}X := (I \rightharpoonup X) \times (O \underset{\text{ne}}{\rightharpoonup} X).$$

non-empty domain!

reverse order

$$(s_i, s_o) \sqsubseteq_{\mathcal{S}X} (t_i, t_o) \quad \overset{\text{def}}{\iff} \quad \underbrace{s_i \sqsubseteq t_i}_{\text{in } I \rightharpoonup X} \text{ and } \underbrace{s_o \sqsupseteq t_o}_{\text{in } O \rightharpoonup X}$$



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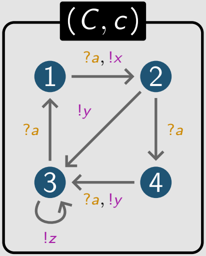
Definition: ioco compatibility relation:

For all $(x, y) \in R$:

- $x \xrightarrow{?i} x'$ and $y \xrightarrow{?i} y' \implies (x', y') \in R$
- $\exists\, o \in O$: $x \xrightarrow{!o} x', y \xrightarrow{!o} y', (x', y') \in R$

van den Bos, Janssen, Moerman '19



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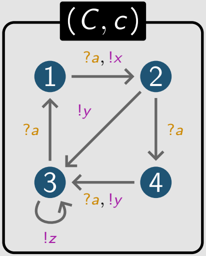

Proposition

ioco compatibility relations = uncertain bisimulations for $\mathcal{S}$

Lax Coalgebra Homomorphisms

Definition: $h\colon (C, c) \xrightarrow{\;\sqsubseteq\;} (D, d)$

$$\begin{array}{ccc} C & \xrightarrow{h} & D \\ {\scriptstyle c}\downarrow & \sqsubseteq & \downarrow{\scriptstyle d} \\ FC & \xrightarrow{Fh} & FD \end{array} \quad \overset{\text{def}}{\iff} \quad \forall x \in C\colon \quad Fh(c(x)) \sqsubseteq_{FX} d(h(x))$$

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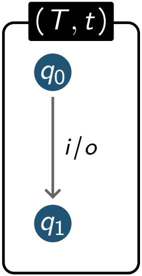

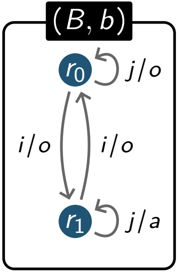

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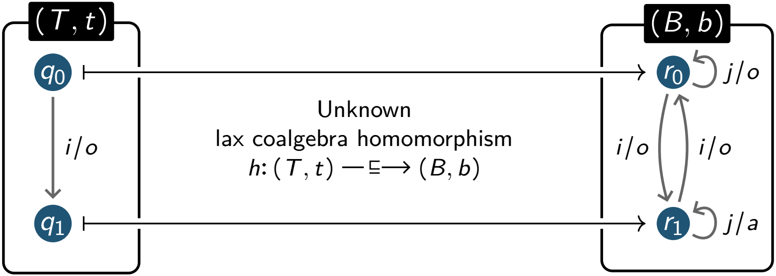

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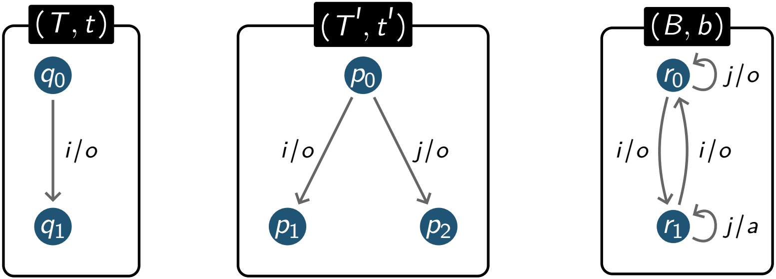

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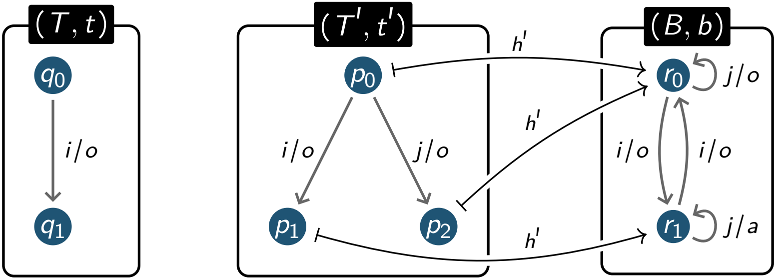
Example $h'\colon (T', t') \xrightarrow{\sqsubseteq} (B, b)$



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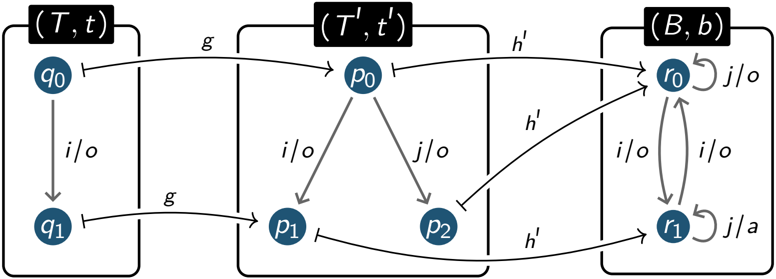
Example $g\colon (T, t) \xrightarrow{\sqsubseteq} (T', t')$ $\quad h'\colon (T', t') \xrightarrow{\sqsubseteq} (B, b)$



Characterization (for F preserving inverse images)

Recall

x, y in $C \xrightarrow{c} FC$ bisimilar $\Longleftrightarrow$ there is some homomorphism $h\colon (C, c) \to (D, d)$ with $h(x) = h(y)$.

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(Half a) Characterization:

x, y in $C \xrightarrow{c} FC$ uncertain bisimilar $\Longleftarrow$ / $\not\Longrightarrow$ there is some lax $h\colon (C, c) \xrightarrow{\sqsubseteq} (D, d)$ with $h(x) = h(y)$.

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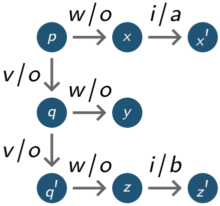
Counterexample for $F := \mathcal{M}$

☺ p, q uncertain bisimilar

☹ $\nexists\ h$ with $h(p) = h(q)$



Characterization via Simulations

Characterization:

x, y in $C \xrightarrow{c} FC$ uncertain bisimilar	$\Longleftarrow$ $\Longrightarrow$	there is $D \xrightarrow{d} FD$, $z \in D$ and a simulation $S \subseteq C \times D$ with $(x, z) \in S$ and $(y, z) \in S$

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Under sufficient conditions:

- For simulations by Hughes, Jacobs:

$$R \quad \mapsto \quad \sqsubseteq \circ \hat{F} R \circ \sqsubseteq$$

- For coalgebraic simulations as arising in from open maps:

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Example:

Reminiscent of existing ioco compatibility characterization in van den Bos, Janssen, Moerman '19

Conclusions

- Definition of *uncertain bisimulation* on F-coalgebras
  … through the lense of relation liftings
- Instances:
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Interesting Properties

- ✔ reflexivity ✔ symmetry ~~transitivity~~
- Uncertain Bisimilarity $\Longleftarrow$ / $\not\Longrightarrow$ Identifiable via a Lax Coalgebra Morphism
- Uncertain Bisimilarity $\Longleftarrow$ / $\Longrightarrow$ Identifiable via a Coalgebraic Simulation

Future

Step towards coalgebraic *automata* learning: Generalized $L^{\#}$

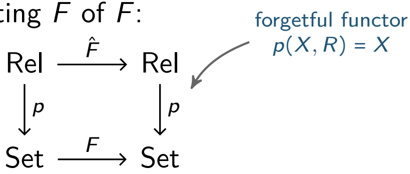
Relation Liftings, Formally

Category Rel of relations

- Objects (X, R): a set X and $R \subseteq X \times X$
- Morphisms $f\colon (X, R) \to (Y, S)$: map $f\colon X \to Y$ with $R \subseteq (f \times f)^{-1}[S]$

Definition: Lifting

Lifting $\hat{F}$ of F:



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