An Efficient Coalgebraic Paige Tarjan Algorithm

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Joint work with:
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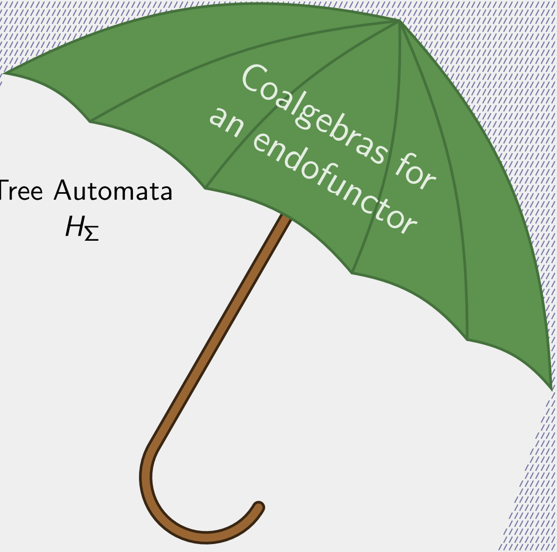
ERLANGEN-NÜRNBERG

Calco Early Ideas
June 16, 2017

Efficient Minimization?



Tree Automata
H_Σ

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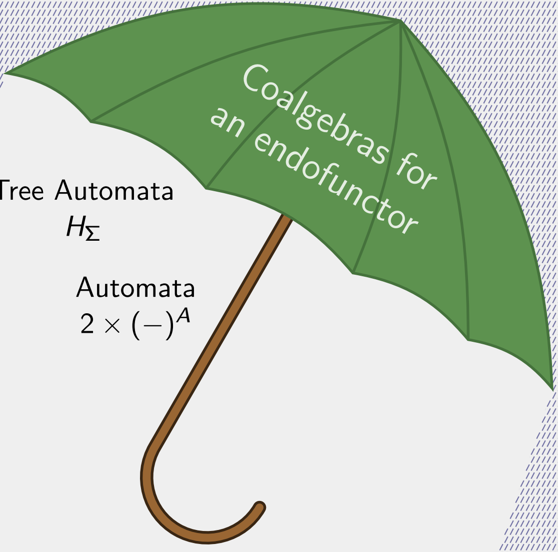

Coalgebras for an endofunctor

Tree Automata
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Automata
$2 \times (-)^A$

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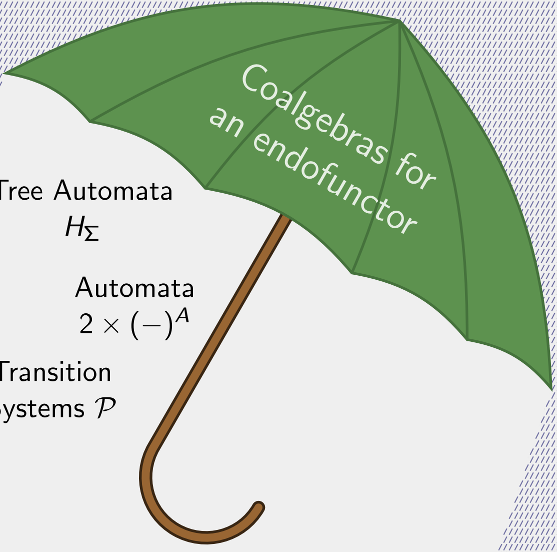

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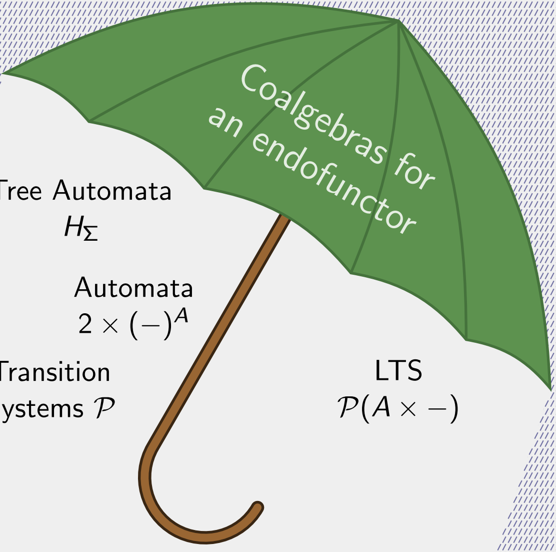

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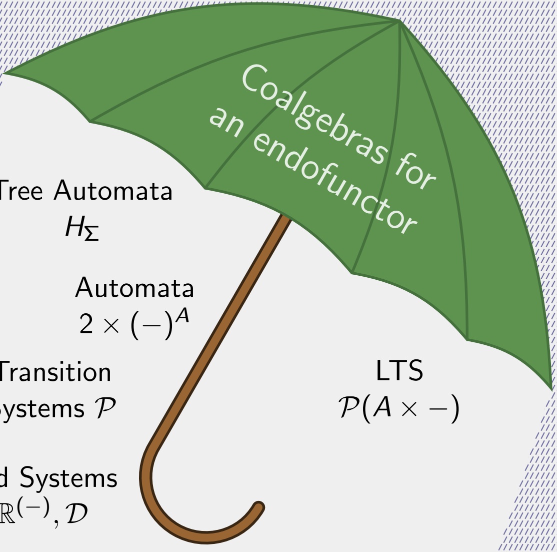

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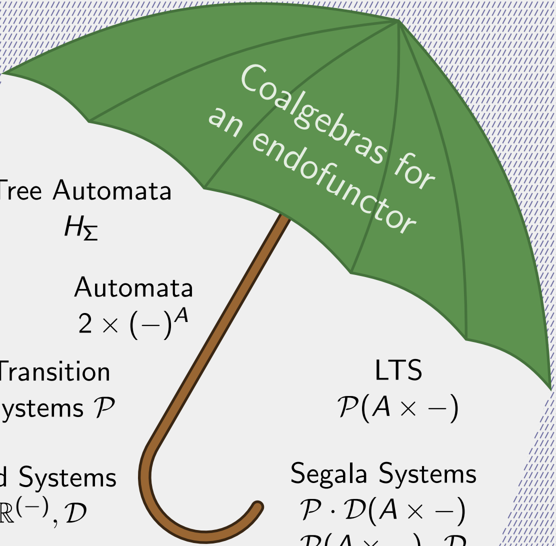

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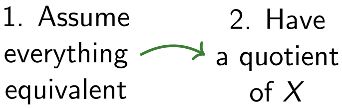

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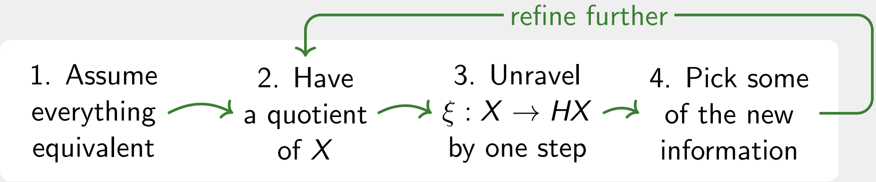

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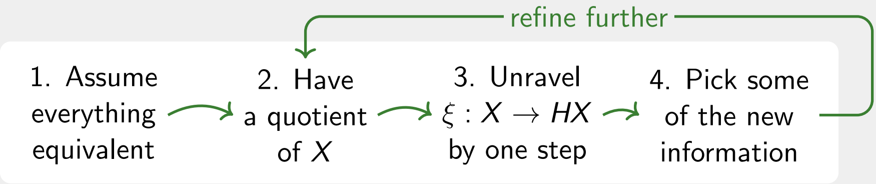

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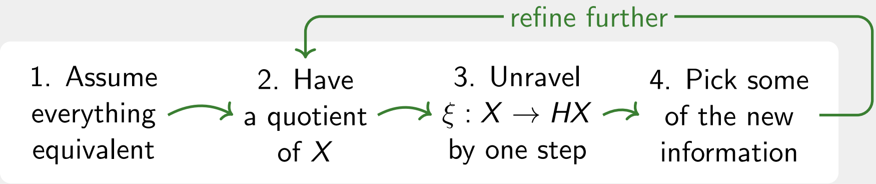
$$\begin{tikzcd} X \arrow[d, "!"'] \\ 1 \end{tikzcd}$$



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 & Q := \ker a \arrow[d, shift left] \arrow[d, shift right] \\
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1 \arrow[ru, bend left] & A
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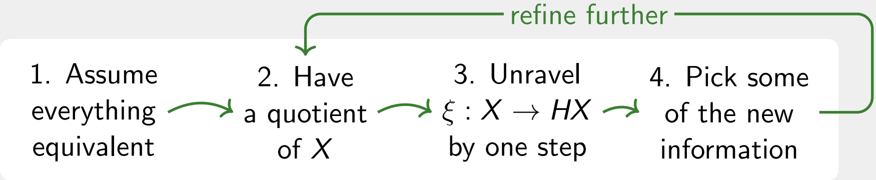

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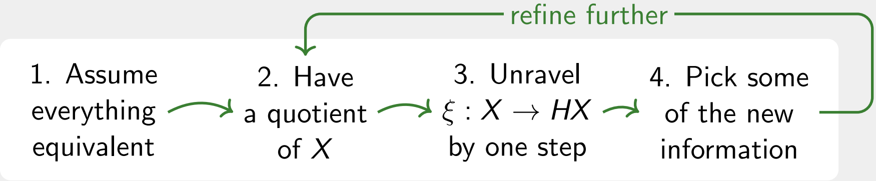

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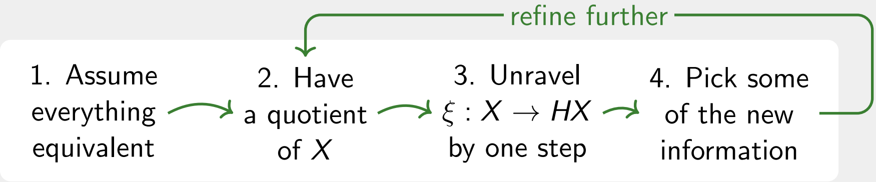
X \arrow[d, "!"'] \arrow[dr, bend left] & & X \arrow[d, "\xi"] \arrow[r, bend left] & X/P \arrow[dd, two heads] \\
1 & X \arrow[d, "a"] \arrow[ur, bend left] & HX \arrow[d, "Ha"] & \\
 & A & HA & X/Q
\end{tikzcd}$$



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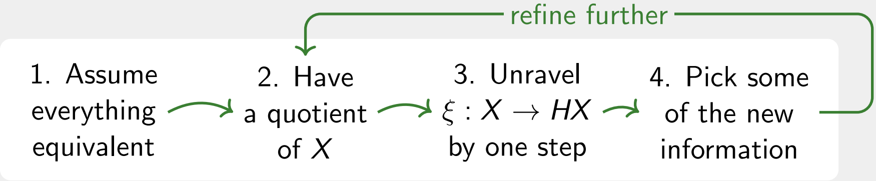
X \arrow[d, "!"'] \arrow[r, bend left] & X \arrow[d, "a"] \arrow[r, bend left] & HX \arrow[d, "Ha"] \arrow[ru, bend left] & X/Q & {} \\
1 & A & HA & {} & {}
\arrow[mapsto, from=D, to=U, bend right, "heuristic"']
\end{tikzcd}$$



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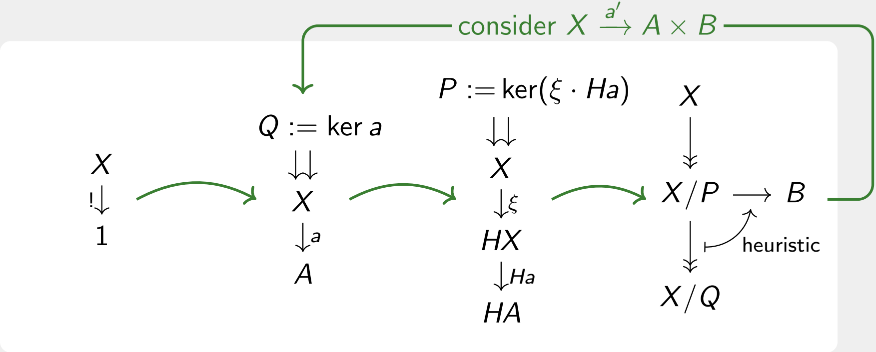

consider $X \xrightarrow{a'} A \times B$

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X \arrow[d, "!"'] \\
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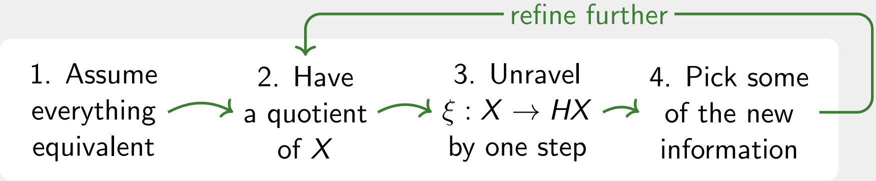
\end{tikzcd}$$

$$\begin{tikzcd}
Q := \ker a \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[d, "a"] \\
A
\end{tikzcd}$$

$$\begin{tikzcd}
P := \ker(\xi \cdot Ha) \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[d, "\xi"] \\
HX \arrow[d, "Ha"] \\
HA
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, two heads] & \\
X/P \arrow[r, ""{name=H}] \arrow[d, two heads, ""{name=V}] & B \\
X/Q &
\arrow[from=V, to=H, mapsto, bend right, "heuristic"']
\end{tikzcd}$$



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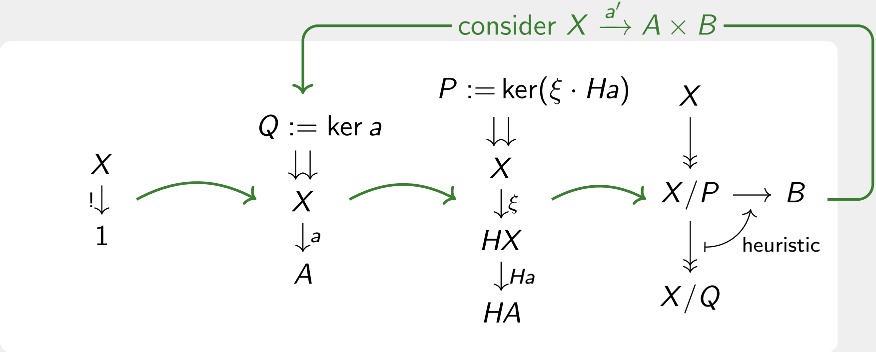

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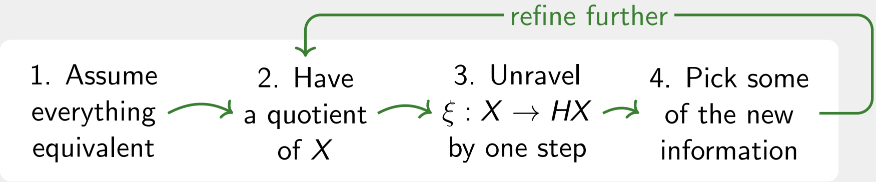
$$\begin{tikzcd}
X \arrow[d, two heads] & \\
X/P \arrow[r, ""{name=R}] \arrow[d, two heads, ""{name=D}] & B \\
X/Q &
\arrow[from=D, to=R, mapsto, bend right, "heuristic"']
\end{tikzcd}$$

Heuristic id on X/P:

Use all immediately



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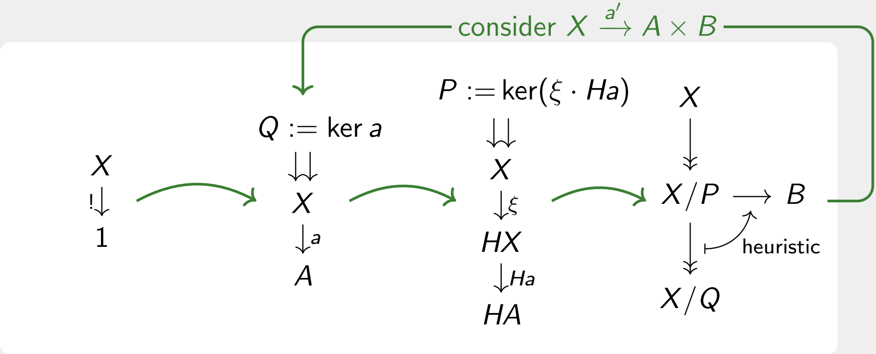

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X \arrow[d, "!"'] \arrow[r, bend left] & X \arrow[d, "a"] \arrow[r, bend left] & HX \arrow[d, "Ha"] \arrow[ru, bend left] & X/Q \arrow[ru, phantom, "\text{heuristic}"] & \\
1 & A & HA & &
\end{tikzcd}$$

Heuristic id on X/P:

Use all immediately

Heuristic in Set:

Process “smaller half”

Assume

Finitely complete, H mono-preserving, (RegularEpi,Mono)-factorisations

Theorem (Correctness)

$$\begin{tikzcd}
X \arrow[r, "\xi"] \arrow[d, two heads] & HX \arrow[d] \\
X/P_i \arrow[r] & H(X/Q_i)
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If $P_i \cong Q_i$, then this

1. is a coalgebra
2. has no proper quotient

Incremental partitions

$$\begin{tikzcd} Q := \ker a \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[d, "a"] \\ A \end{tikzcd}$$

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$$\begin{tikzcd}
Q := \ker a \arrow[d, shift left] \arrow[d, shift right] & & Q \cap \ker b \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[d, "a"] \arrow[rr, bend left] & & X \arrow[d, "\langle a,b \rangle"] \\
A & & A \times B
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Question: When is $\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle$?

Incremental partitions

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Theorem: In Set, $\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle$ if

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Theorem: In Set, $\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle$ if

$$\begin{tikzcd}
H(L+R) \arrow[d] \\
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monic and

“zippable”

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Category:	Heuristic:	Functor:
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Assumption: Functor encoding

- coalgebra structure as edges with labels

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- compute “smaller half” heuristic in linear time

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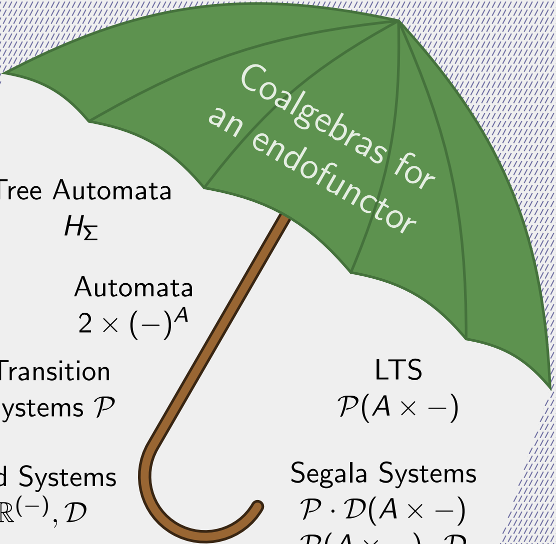
Theorem

Overall complexity: $\mathcal{O}((m+n)\cdot \log n)$ for $n = |X|$, $m = \sum_{x \in X} |\flat\xi(x)|$



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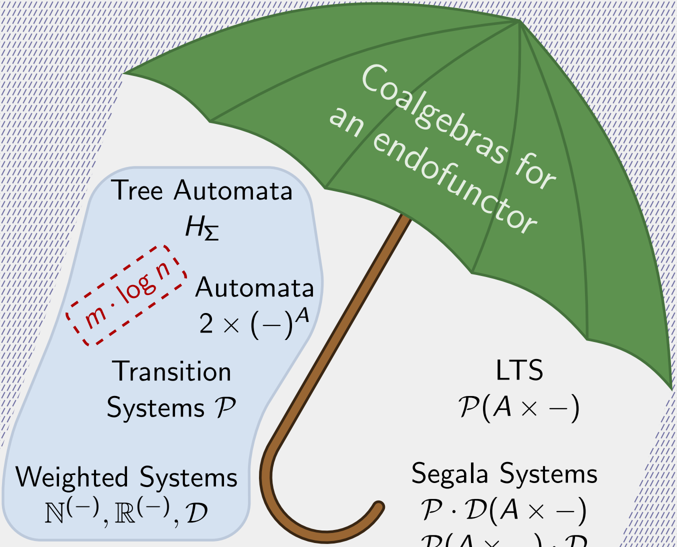

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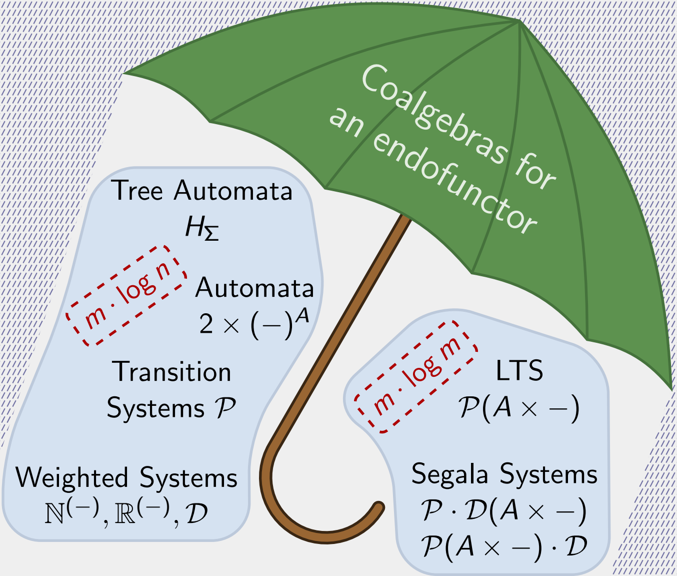

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Efficient Minimization

Functors H zippable, if

$$H(L+R) \xrightarrow{\text{unzip}} H(L+1)\times H(1+R) \text{ is monic.}$$

E.g. Id, Constants, $\times$, $+$, $\hookrightarrow$, $M^{(-)}$, part. additive

Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{\_\}$

$$\begin{tikzcd}
a_1\, a_2\, b_1\, a_3\, b_2 \arrow[d, mapsto, "\text{unzip}"] \\
(a_1 a_2\ \_\ a_3\ \_,\ \ \_\ \_\ b_1\ \_\ b_2)
\end{tikzcd}$$

$(-)^*$ is zippable

$$\begin{tikzcd}
\{a_1, a_2, b_1\} \arrow[d, mapsto, "\text{unzip}"] \\
(\{a_1, a_2, \_\},\ \{\_, b_1\})
\end{tikzcd}$$

$\mathcal{P}$ is zippable

Functors H zippable, if

$$H(L+R) \xrightarrow{\text{unzip}} H(L+1) \times H(1+R) \text{ is monic.}$$

E.g. Id, Constants, $\times$, $+$, $\hookrightarrow$, $M^{(-)}$, part. additive

Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{\_\}$

$$a_1\, a_2\, b_1\, a_3\, b_2 \xmapsto{\text{unzip}} (a_1 a_2 \,\_\, a_3 \,\_,\ \_\,\_\, b_1 \,\_\, b_2)$$

$(-)^*$ is zippable

$$\{a_1, a_2, b_1\} \xmapsto{\text{unzip}} (\{a_1, a_2, \_\},\ \{\_, b_1\})$$

$\mathcal{P}$ is zippable

$$\{\{a_1, b_1\}, \{a_2, b_2\}\} \xmapsto{\text{unzip}} (\{\{a_1, \_\}, \{a_2, \_\}\},\ \{\{\_, b_1\}, \{\_, b_2\}\}) \xleftarrow{\text{unzip}} \{\{a_1, b_2\}, \{a_2, b_1\}\}$$

$\mathcal{P}\mathcal{P}$ is not zippable

~~Composition~~

~~Quotients~~

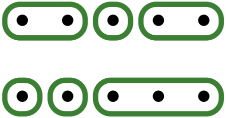
$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

$\Leftrightarrow$ $\ker a \cup \ker b$ transitive

$\Leftrightarrow$ $\forall x \in X : [x]_a \subseteq [x]_b$ or $[x]_a \supseteq [x]_b$

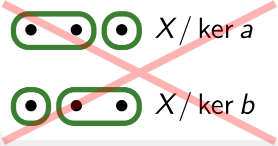
Example



$X / \ker a$

$X / \ker b$

Non-Example



$X / \ker a$

$X / \ker b$

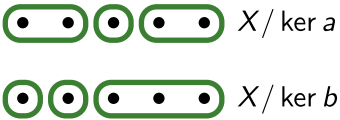
$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

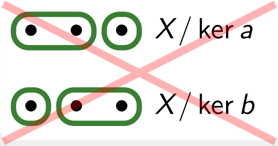
$\Leftrightarrow$ $\ker a \cup \ker b$ transitive

$\Leftrightarrow$ $\forall x \in X : [x]_a \subseteq [x]_b$ or $[x]_a \supseteq [x]_b$

Example



Non-Example



Process smaller half for $X \overset{f}{\twoheadrightarrow} F \overset{g}{\twoheadrightarrow} G$

Find $x \in X$, with $S := [x]_f$, $C := [x]_{gf}$, such that $2 \cdot |S| \leq |C|$.
Return $\langle \chi_S, \chi_C \rangle : X \to 2 \times 2$

Functor encoding

- internal weights W, $w : HX \to \mathcal{P}X \to W$
- edge labels L
- $\flat : HX \to \mathcal{B}_{\mathrm{f}}(L \times X)$
- update : $\underset{\substack{\uparrow \\ \text{edges into } S}}{\mathcal{B}_{\mathrm{f}}(L)} \times \underset{\substack{\uparrow \\ \text{weight of } C}}{W} \longrightarrow \underset{\substack{\uparrow \\ \text{weight of } S}}{W} \times \underset{\substack{\uparrow \\ H\langle \chi_S, \chi_C \rangle}}{H(2 \times 2)} \times \underset{\substack{\uparrow \\ \text{weight of } C \setminus S}}{W}$

Functor:	$G^{(-)}$	$\mathcal{B}_{\mathrm{f}}$	$\mathcal{D}$	$\mathcal{P}$	H_Σ
Labels L:	G	$\mathbb{N}$	$[0,1]$	1	$\mathbb{N}$
Weights W:	$G^{(2)}$	$\mathcal{B}_{\mathrm{f}}2$	$\mathcal{D}2$	$\mathbb{N}$	$H_\Sigma 2$
$w(C)$, $C \subseteq Y$:	$G\chi_C$	$\mathcal{B}_{\mathrm{f}}\chi_C$	$\mathcal{D}\chi_C$	$\lvert C \cap (-) \rvert$	$H_\Sigma \chi_C$

Future work

- Implementation & Benchmarking
- $\mathcal{O}(m \cdot \log n)$ on $\mathcal{P}(A \times -)$
- $\ker\langle Ha, Hb\rangle = \ker H\langle a, b\rangle$ outside of Set?
- Further functors, e.g. monotone neighbourhoods.