Supported Sets

A New Foundation For Nominal Sets And Automata

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CSL, Warsaw, Feb 13, 2023

Motivation 1

Nominal Sets

- ☺ Freshness & Binding of Names
- ☺ Initial Algebras: e.g. Lambda-Expressions mod. $\equiv_\alpha$
- ☺ (Final) Coalgebras:
  - Infinite Lambda-Trees (mod. $\equiv_\alpha$)
  - Automata with name binding
- ☺ Rich categorical structure: boolean topos

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Drawbacks

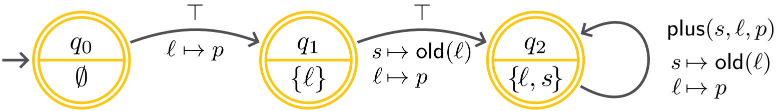
- ⚡ finitely presentable = orbit-finite $\neq$ finite
- ⚡ not monadic over Set

Eilenberg-Moore category for a monad

Motivation 2

Register Automata (RA) & Data Languages

Finite Automata for Infinite Input Alphabets



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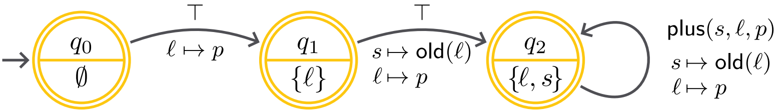

Set-Theoretic RA

- ☹ long definition
- ☹ ad-hoc for purpose
- ☺ obviously finite description

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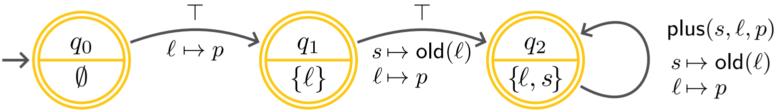

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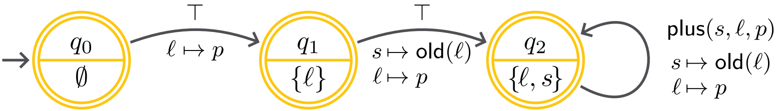

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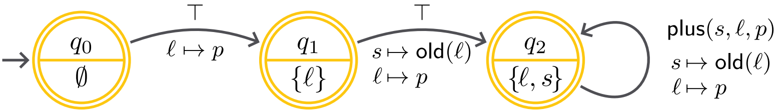

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? Formal Relationship ?

Definition: The category $\mathsf{Supp}(A)$ (for a fixed set A)

- supported set = set X with a map $\mathsf{s}_X \colon X \to \mathcal{P}_{\mathrm{f}}(A)$

Idea

- Every $x \in X$ holds finitely many elements $\mathsf{s}_X(x) \subseteq A$ in registers.

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Examples Objects

- the set A itself, $\mathsf{s}_A(a) = \{a\}$
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- pairs A^2, $\mathsf{s}(a, b) = \{a, b\}$

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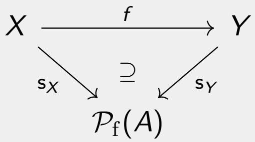
- Maps f may clear registers but can't invent new values.



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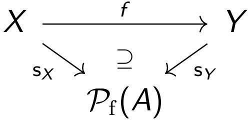
Categorical Properties of Supp(A)



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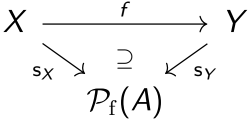

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Every supported set is a union of singleton supported sets

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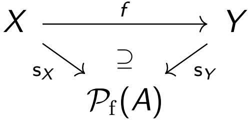

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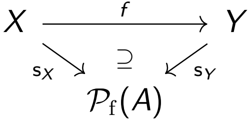

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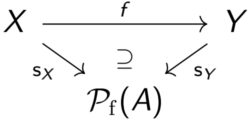

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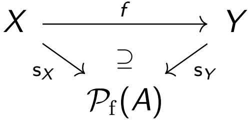

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$JX = X$, $s_{JX}(x) = \emptyset$

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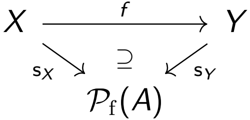

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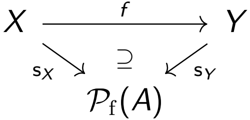

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$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_{s_X}\searrow \quad \supseteq \quad \swarrow_{s_Y} & \\ & \mathcal{P}_{\mathrm{f}}(A) & \end{array}$$

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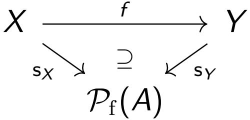

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$f\colon X \to Y$ support reflecting $\quad :\Leftrightarrow \quad s(f(x)) = s(x) \quad \forall x \in X$

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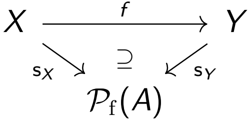

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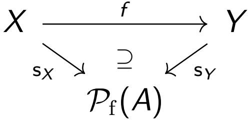

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$f\colon X \to Y$ is injective and support reflecting

$\Leftrightarrow$ there is χ s.t. $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi} \\ 1 & \longrightarrow & 2 \end{array}$ is a pullback

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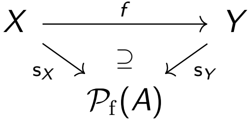

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- ☺ is a quasitopos (but not a topos ☹)

Name Binding Functor via de Bruijn indices

$\mathcal{B}\colon \mathsf{Supp}(\mathbb{A}) \to \mathsf{Supp}(\mathbb{A})$ when assuming $A := \mathbb{A} = \{a_0, a_1, \ldots\}$

$$\mathcal{B}X = X \qquad \mathsf{s}_{\mathcal{B}X}(x) := \{a_k \mid a_{k+1} \in \mathsf{s}_X(x), k \in \mathbb{N}\}$$

Index = Number of other binders inbetween

$$\lambda x.\ x\ (\lambda x.\ x)\ (\lambda y.\ x\ y) \iff \lambda.\ a_0\ (\lambda.\ a_0)\ (\lambda.\ a_1\ a_0)$$

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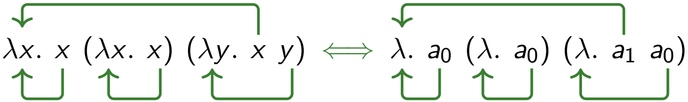
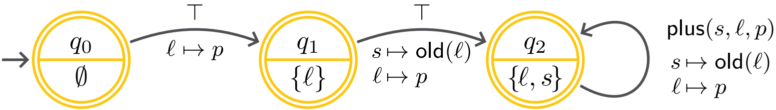


RA by Cassel, Howar, Jonsson, Steffen '18 correspond to:

Finite Q and $\delta\colon Q \to 2 \times \mathcal{B}\mathcal{P}_{\mathrm{f}}(\Sigma\mathbb{A} \times TQ)$ in $\mathsf{Supp}(\mathbb{A})$



M-sets for a submonoid $M \subseteq (A \to A, \circ, \mathrm{id}_A)$

- M-set = set X with $\cdot\colon M \times X \to X$ (+ axioms)
- homomorphism = map $f\colon X \to Y$ with $f(m \cdot x) = m \cdot f(x)$

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Support (informally)

- $S \subseteq A$ supports $x \in X \Leftrightarrow m \cdot x$ is determined by $m(a)$, $a \in S$
- $\mathrm{supp}(x)$ $=$ least **finite** $S \subseteq A$ which supports x (if it exists)

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Nominal M-sets

M-sets $(X, \cdot)$ in which all elements have a (least) finite support.

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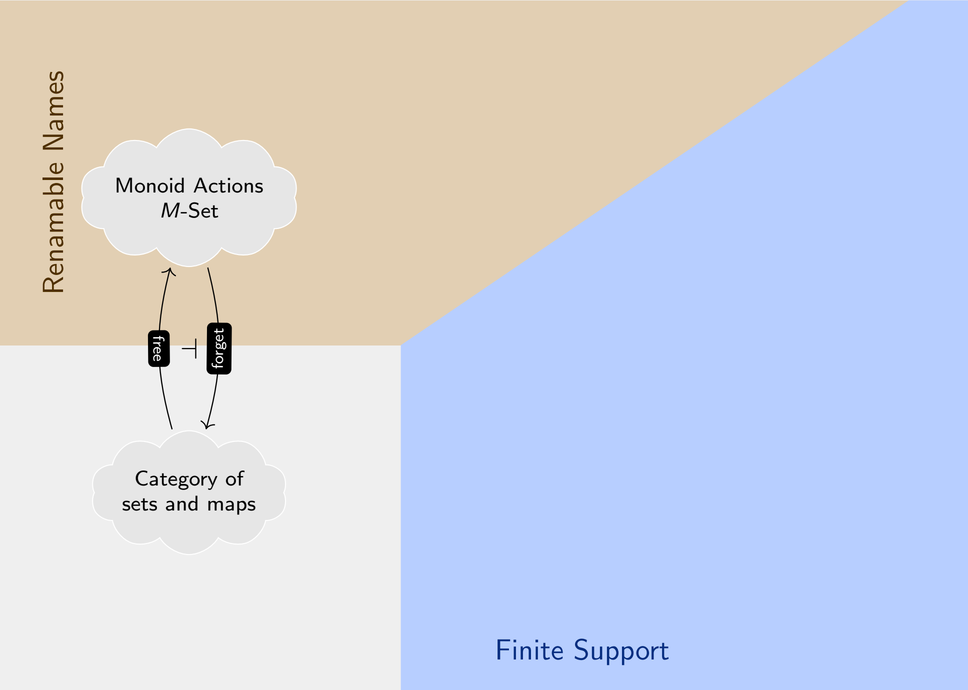
Forgetful functor $U : \mathsf{Nom}(M) \to \mathsf{Supp}(A)$

$U(X, \cdot) = X$ with $\mathsf{s}_X(x) := \mathsf{supp}_X(x)$

Renamable Names

Category of sets and maps

Finite Support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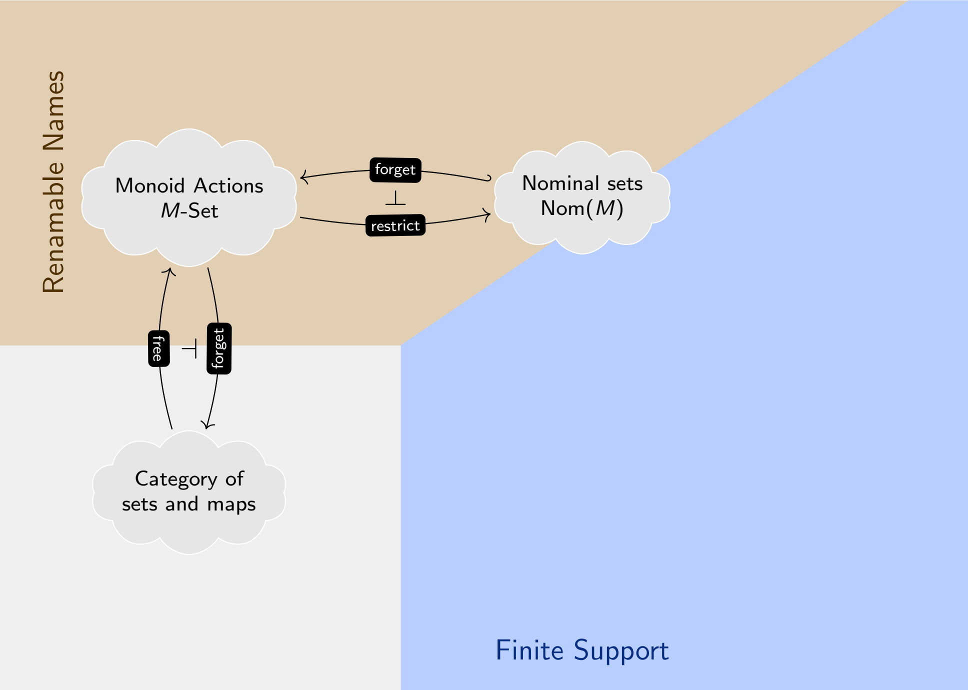
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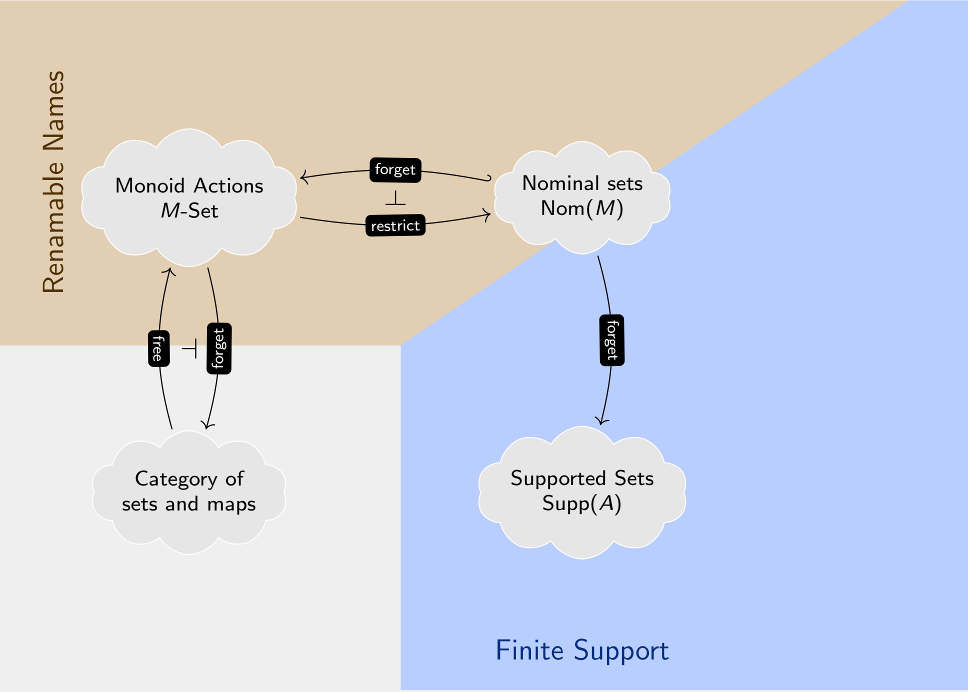
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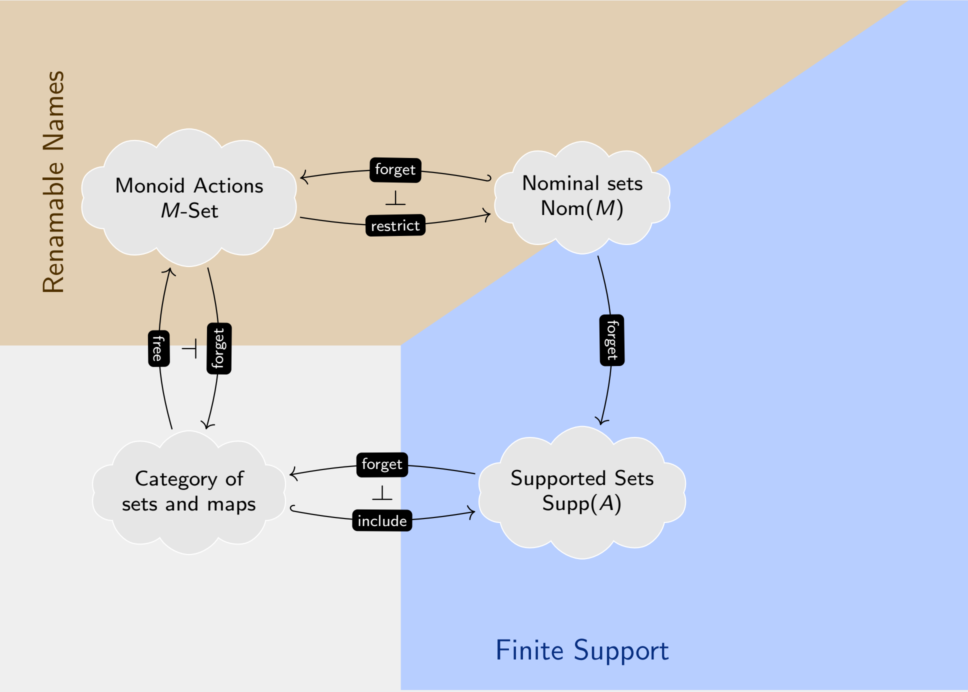
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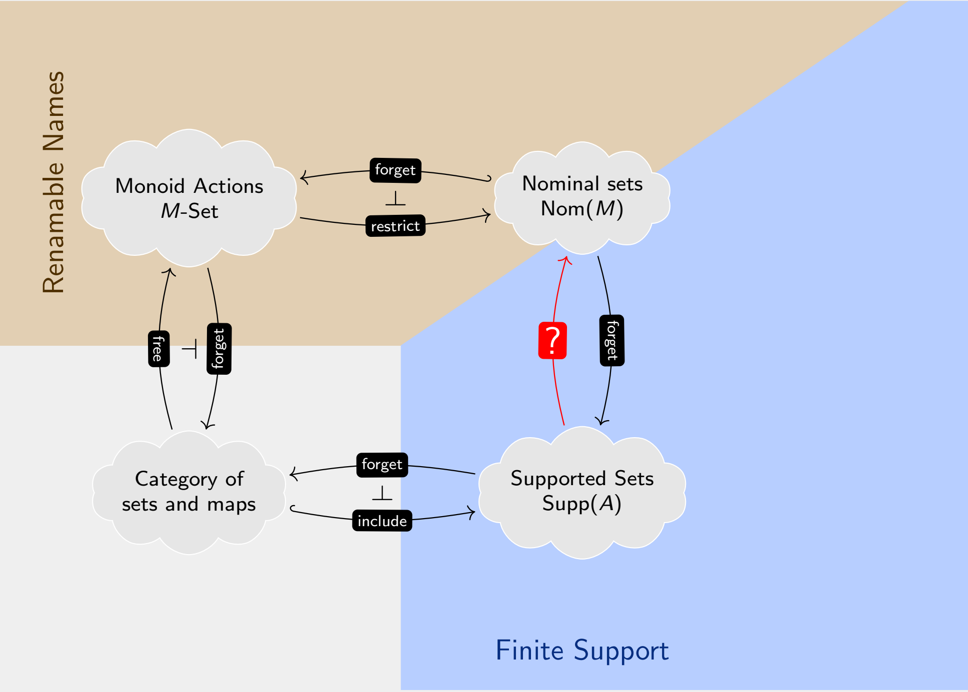
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Finite Support

Monads T

- Correspond to algebraic theories (Groups, Rings, ...)
- Models = Eilenberg-Moore category $\mathcal{EM}(T)$
  - ☺ Free/universal constructions in $\mathcal{EM}(T)$
  - ☺ Plethora of generic methods applicable
- Adjunction is monadic $\Leftrightarrow$ $\mathcal{D} = \mathcal{EM}(T)$, $R \cdot L = T$

$$\mathcal{D} \quad L \left(\dashv \right) R \quad \mathcal{C}$$

☺ Monoid actions M-Set is monadic over Set

☹ Nom is not monadic over Set

Theorem

For all $M \subseteq A^A$, the functor $U\colon \mathsf{Nom}(M) \to \mathsf{Supp}(A)$ is

- right-adjoint if M admits least supports

Instances:

Order Symmetry	*Equality Symmetry*	*Renaming Sets*
$\mathsf{Nom}(\mathsf{Aut}(\mathbb{Q}, <))$	$\mathsf{Nom}(\mathsf{Perm}_\mathsf{f}(\mathbb{A}))$	$\mathsf{Nom}(\mathsf{Fin}(\mathbb{A}))$

Theorem

For all $M \subseteq A^A$, the functor $U\colon \mathsf{Nom}(M) \to \mathsf{Supp}(A)$ is

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if $\forall R \in \mathcal{P}_{\mathrm{f}} A,\ a \notin R\colon \exists \ell \in M\colon\ \ell(a) \neq a,\ \ell(r) = r\ \forall r \in R$

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$TX = \{(m, x) \mid m\colon \mathsf{s}(x) \rightarrowtail \mathbb{A}, x \in X\}$ for $M := \mathrm{Perm}_{\mathrm{f}}(\mathbb{A})$

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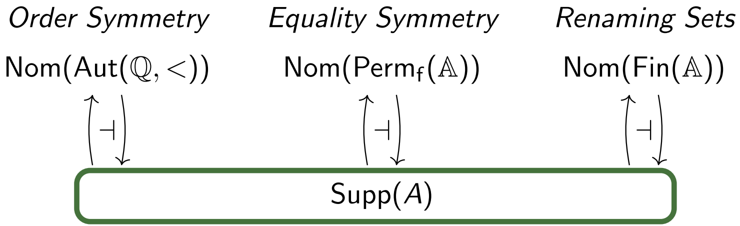
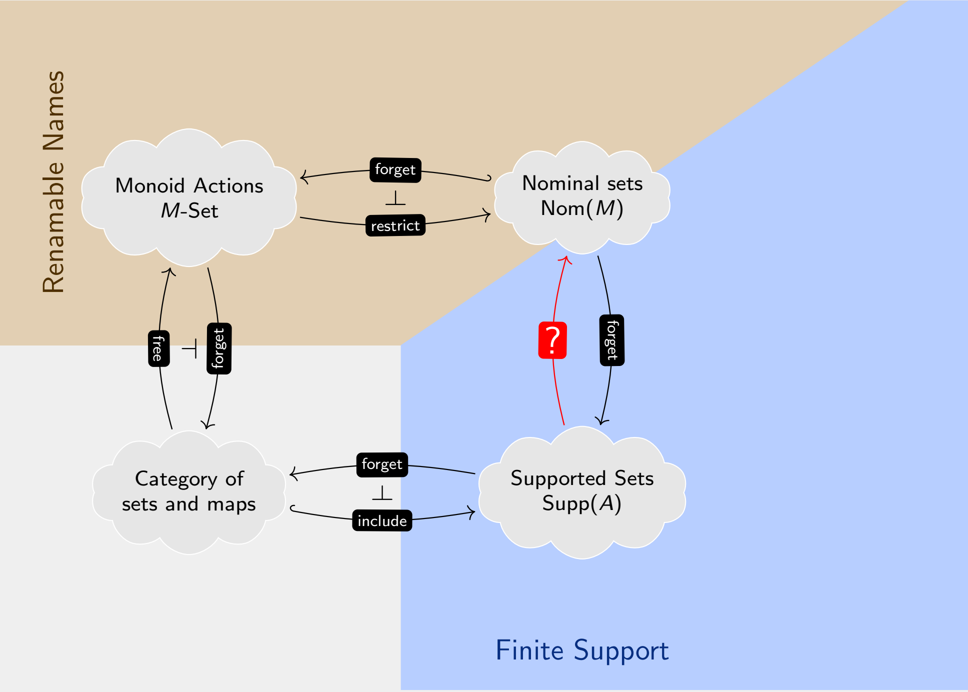
Instances: Monadic adjunctions (A countably infinite)

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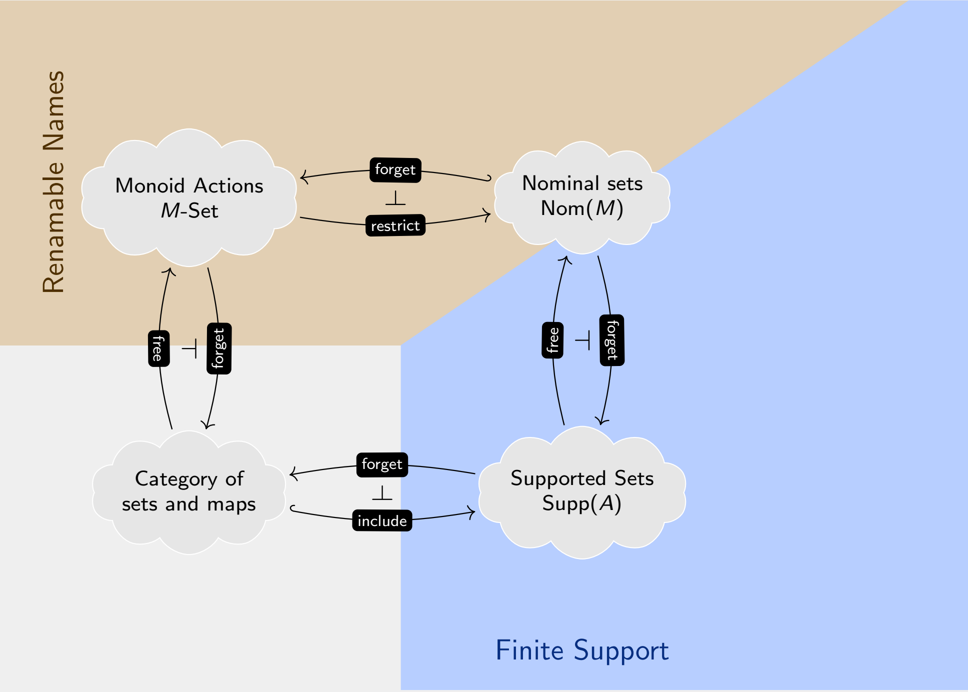
Finite Support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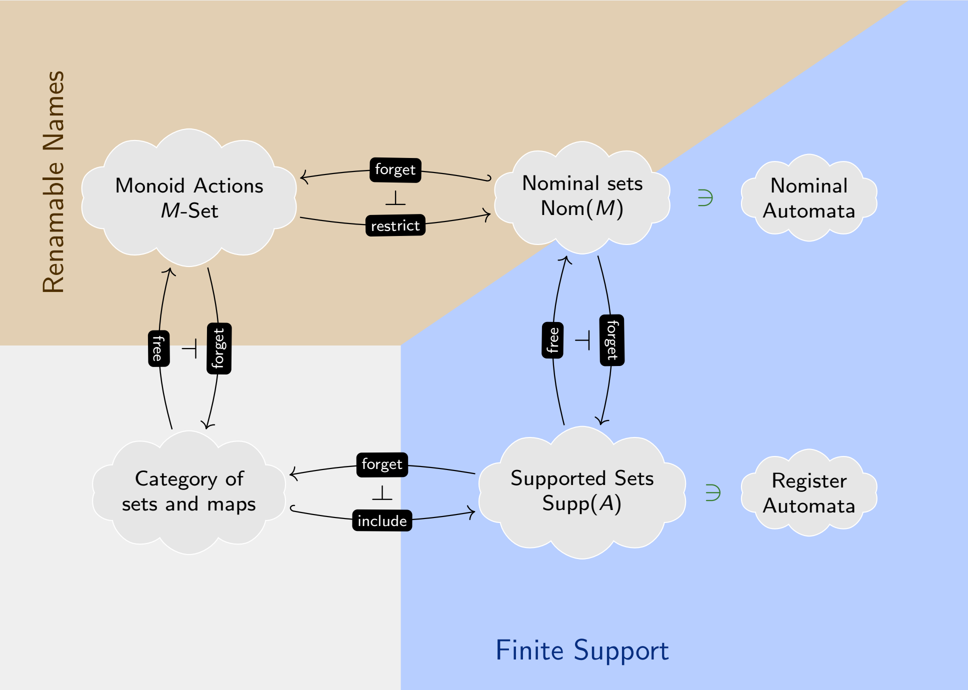
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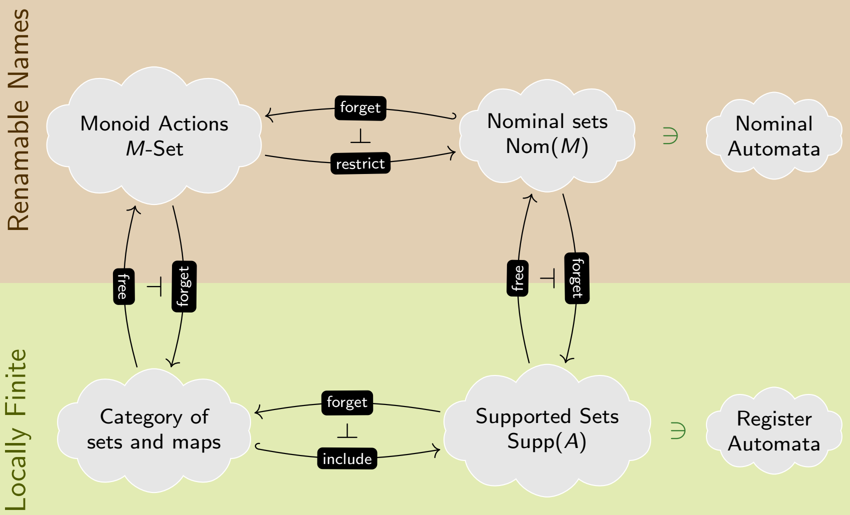
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Application: finite representation of nominal M-sets

nominal set orbit-finite	$\Longleftrightarrow$	finite supported set G with finitely many equations $E \subseteq TG \times TG$

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Application: Internalize Side-Effects

$$\begin{array}{ccc} \mathrm{Nom}(M) & \xrightarrow{\bar{H}} & \mathrm{Nom}(M) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Supp}(A) & \xrightarrow{H} & \mathrm{Supp}(A) \end{array} \quad\Longrightarrow\quad \begin{array}{c} \mathrm{Coalg}(\bar{H}) \\ \uparrow{\scriptstyle \text{internalize side effects}} \\ \mathrm{Coalg}(HT) \end{array}$$

$\mathcal{EM}(T)$ → $\mathrm{Nom}(M)$

Configuration Space ($\mathrm{Coalg}(\bar{H})$)

State Space ($\mathrm{Coalg}(HT)$)

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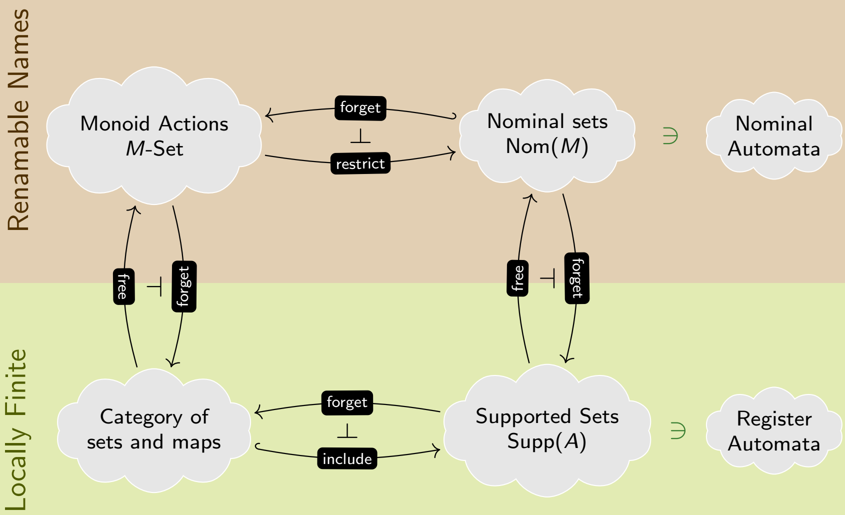
Configuration Space

State Space

Functors that lift:

- $\times$, $+$, $\mathcal{P}_{\mathrm{f}}$, $\mathcal{P}_{\mathrm{ufs}}$, ...
- Binding functor $\mathcal{B}$ (for $A := \mathbb{A}$) ← Lifts to Gabbay & Pitts' abstraction functor $[\mathbb{A}]$
- E.g. RNNA $HX := 2 \times \mathcal{P}_{\mathrm{ufs}}(\mathcal{B}X + \mathbb{A} \times X)$
- E.g. Register Automata $HX := 2 \times \mathcal{B}\mathcal{P}_{\mathrm{f}}(\Sigma\mathbb{A} \times X)$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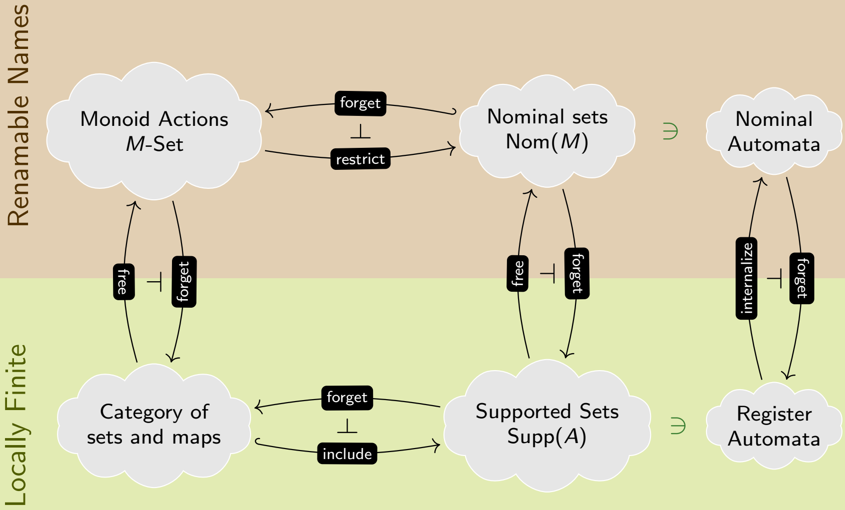
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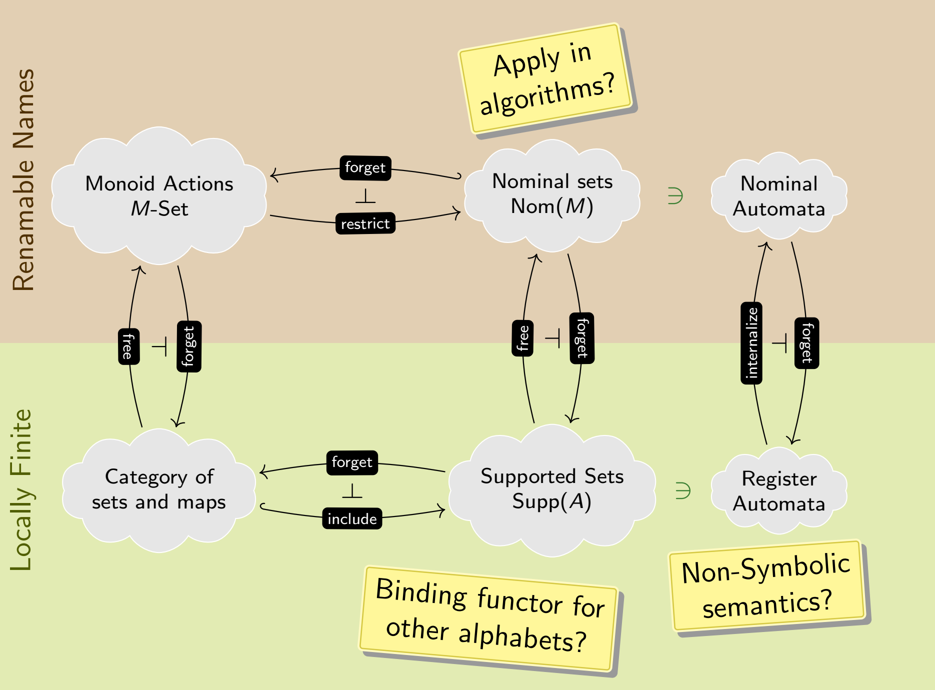
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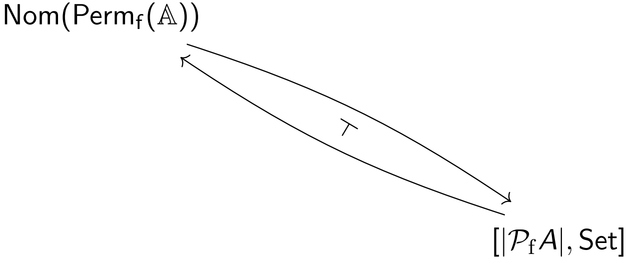
Register Automata
Apply in algorithms?
Binding functor for other alphabets?
Non-Symbolic semantics?

Relation to Presheaves



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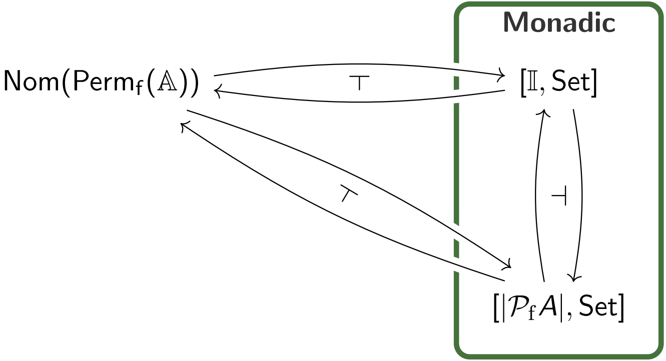
Relation to Presheaves

$\mathbb{I} = \mathcal{P}_{\mathrm{f}}(\mathbb{A})$ with all injective maps



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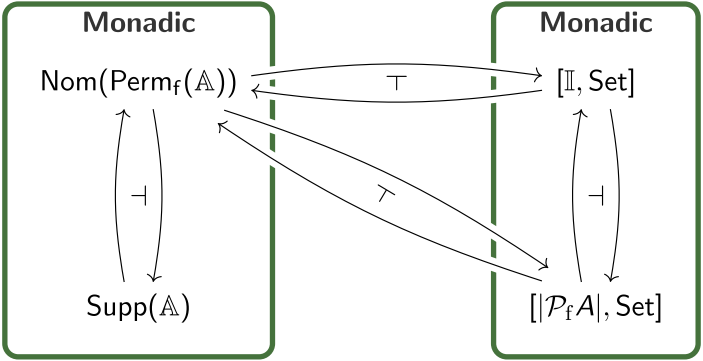

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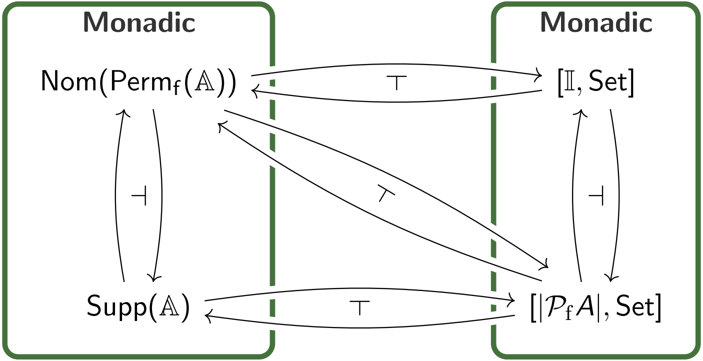

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