The Locally Finite Fixpoint

A uniform framework for finite state and equation systems

Stefan Milius, Dirk Pattinson, **Thorsten Wißmann**

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Australian National University

CALCO 2015 Early Ideas

June 26, 2015

Old Approach: The Rational Fixpoint

E.g. $HX = 2 \times X^{\Sigma}$

(C, c) — Deterministic Automata

Rational Fixpoint → $(\varrho H, r)$ — Regular Languages

$$(\varrho H, r) \xrightarrow{r^\dagger} (\nu H, \tau)$$

$(\nu H, \tau)$ — All Formal Languages

Adámek, Milius, Velebil ’06

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E.g. $HX = 2 \times X^{\Sigma}$

C finitely presentable — (C, c) — Finite Deterministic Automata

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finitely presentable objects $\subseteq$ finitely generated objects

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$\downarrow r^{\dagger}$ ← Not monic in general!

$(\nu H, \tau)$ — All Formal Languages

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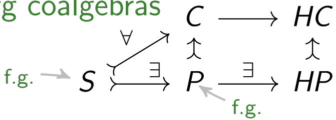
finitely presentable objects $\subseteq$ finitely generated objects

New Approach via Finitely Generated (f.g.) Objects

Coalgebras with f.g. carrier

$C \longrightarrow HC$

lfg coalgebras



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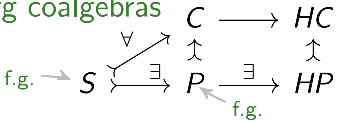

Theorem: The Locally Finite Fixpoint of H

Suppose $H : \mathcal{C} \to \mathcal{C}$ mono-preserving and finitary endofunctor.
Final locally finitely generated (lfg) coalgebra ϑH exists and is

- subcoalgebra of νH.
- an isomorphism.

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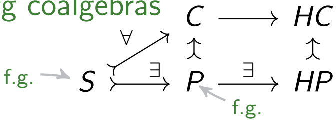

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The Inverse: Initial fg-iterative algebra

(A, α) fg-iterative $\Leftrightarrow$ solves $X \to HX + A$ (X f.g.) uniquely.

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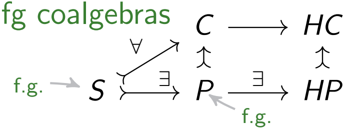

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Under slightly stronger assumptions

$$\varrho H \twoheadrightarrow \vartheta H \rightarrowtail \nu H$$

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton → $x : X \to HTX$ in Set

$$\frac{x : X \to HTX \text{ in Set}}{x^\sharp : TX \to \bar{H}TX \text{ in } \mathsf{Set}^T}$$

its determinization → $x^\sharp : TX \to \bar{H}TX$ in Set^T

Computational side effect → T

Silva, Bonchi, Bonsangue, Rutten '13

Proposition

$$\vartheta\bar{H} = \bigcup_{\substack{x:X\to HTX\\ x \text{ finite}}} \mathrm{Im}(x^{\sharp\dagger}) \subseteq \nu H$$

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Grammars in weak Greibach NF

$T = \mathcal{P}_\mathrm{f}((- + \Sigma)^*) \quad H = 2 \times (-)^\Sigma$

Winter, Bonsangue, Rutten '13

$\Rightarrow \vartheta \bar{H} =$ context-free languages.

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Stack Machines

$T =$ Stack-Monad Goncharov '13

$\otimes$ non-determinism

Goncharov, Milius, Silva '14

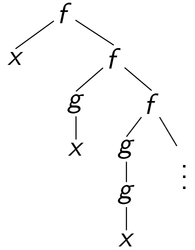
$\Rightarrow \vartheta\bar{H} \approx$ context-free languages.

Algebraic Trees & Recursive Program Schemes

Example: For the signature $\Sigma = \{{}^{f}/_{2}, {}^{g}/_{1}\}$

$$\varphi(x) = f(x, \varphi(g(x)))$$

Solution of $\varphi(x)$



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Coalgebraically

f.p.$\subsetneq$f.g.

$\mathcal{C}$ = finitary H_Σ-pointed Set-Monads:

RPS → $e : B \longrightarrow H_\Sigma \cdot B + \mathrm{Id}$

Adámek, Milius, Velebil '11

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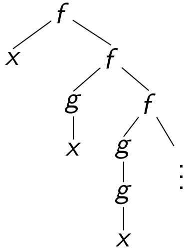

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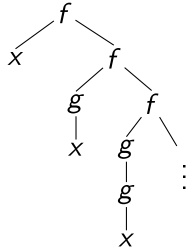

Proposition

Monad of algebraic trees $= \vartheta(H_\Sigma \cdot (-) + \mathsf{Id})$

Further details

tinyurl.com/coalgebra

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www8.cs.fau.de/thorsten

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