A New Foundation for Finitary Corecursion

The Locally Finite Fixpoint and its Properties

Stefan Milius, Dirk Pattinson, **Thorsten Wißmann**

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Australian National University

FoSSaCS 2016

April 4, 2016

Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

$$C \xrightarrow{c} HC$$

Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

$$C \xrightarrow{c} HC$$

E.g. $HX = 2 \times X^{\Sigma}$: $\quad \mathsf{Set} \to \mathsf{Set}$

Coalgebras

States, variables

Successor type, Signature

Carrier

Functor

$$C \xrightarrow{c} HC$$

E.g. $HX = 2 \times X^{\Sigma}$: $\mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra: } C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Deterministic Automata

Coalgebras

States, variables

Successor type, Signature

E.g. $HX = 2 \times X^{\Sigma}$: $\mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra: } C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Carrier

Functor

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, dashed, "c^{\dagger}"'] & HC \arrow[d, "Hc^{\dagger}"] \\
\nu H \arrow[r, "\tau", "\cong"'] & H\nu H
\end{tikzcd}$$

Deterministic Automata

All Formal Languages

unique

solution, semantics

Final Coalgebra

Coalgebras

States, variables

Successor type, Signature

E.g. $HX = 2 \times X^{\Sigma}$: $\mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra: } C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Carrier

Functor

C f.p.

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, dashed, "c^{\dagger}"'] & HC \arrow[d, "Hc^{\dagger}"] \\ \nu H \arrow[r, "\tau", "\cong"'] & H\nu H \end{tikzcd}$$

Finite Deterministic Automata

unique

solution, semantics

All Formal Languages

Final Coalgebra

Coalgebras

States, variables

Successor type, Signature

E.g. $HX = 2 \times X^{\Sigma}$: $\quad \mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra: } C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Carrier

Functor

C f.p. $\quad C \xrightarrow{c} HC \qquad$ Finite Deterministic Automata

Rational Fixpoint $\quad \varrho H \xrightarrow[\cong]{r} H\varrho H$

$\nu H \xrightarrow[\cong]{\tau} H\nu H \qquad$ All Formal Languages

Final Coalgebra

Adámek, Milius, Velebil'06

Coalgebras

States, variables

Successor type, Signature

E.g. $HX = 2 \times X^{\Sigma}$: $\mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra: } C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Carrier

Functor

C f.p.

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, dashed, "c^{+}"'] & HC \arrow[d, "Hc^{+}"] \\
\varrho H \arrow[r, "r", "\cong"'] & H\varrho H
\end{tikzcd}$$

Finite Deterministic Automata

Rational Fixpoint

Regular Languages

$$\nu H \xrightarrow[\cong]{\tau} H\nu H$$

All Formal Languages

Final Coalgebra

Adámek, Milius, Velebil'06

Coalgebras

States, variables

Successor type, Signature

E.g. $HX = 2 \times X^{\Sigma}$: $\quad \mathsf{Set} \to \mathsf{Set}$

$$\frac{H\text{-coalgebra:}\ C \to 2 \times C^{\Sigma}}{C \to 2 \quad \& \quad C \times \Sigma \to C}$$

Carrier

Functor

C f.p.

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, dashed, "c^{+}"'] & HC \arrow[d, "Hc^{+}"] \\
\varrho H \arrow[r, "r", "\cong"'] \arrow[d, dashed, "r^{\dagger}"'] & H\varrho H \arrow[d, "Hr^{\dagger}"] \\
\nu H \arrow[r, "\tau", "\cong"'] & H\nu H
\end{tikzcd}$$

Rational Fixpoint

Final Coalgebra

Finite Deterministic Automata

$$\begin{tikzcd}
\text{Regular Languages} \arrow[d, hook] \\
\text{All Formal Languages}
\end{tikzcd}$$

Adámek, Milius, Velebil'06

The Rational Fixpoint

Previous work

- Categorical approach to iteration theories by Bloom & Ésik
  Universal Property as an algebra.

  Adámek, Milius, Velebil (MSCS'06)

The Rational Fixpoint

Previous work

- Categorical approach to iteration theories by Bloom & Ésik
  Universal Property as an algebra.
  Adámek, Milius, Velebil (MSCS'06)
- Universal Property as coalgebra — Milius (LICS'10)

The Rational Fixpoint

Previous work

- Categorical approach to iteration theories by Bloom & Ésik
  Universal Property as an algebra.
  Adámek, Milius, Velebil (MSCS'06)
- Universal Property as coalgebra Milius (LICS'10)
- Syntax & Axiomatization: Generalized Regular Expressions
  Silva, Bonchi, Bonsangue, Rutten (LICS'09, CONCUR'09)
  Bonsangue, Milius, Silva (TOCL'11)

Different Parameters for $\mathcal{C}$ and H

Instances of ϱH

- Rational Σ-trees for a signature Σ (on Set)
- Eventually periodic streams (on Set)
- Rational streams (on Vec_K) — Milius (LICS'10)
- Rational λ-trees (on $\mathsf{Set}^{\mathscr{F}}$) — Adámek, Milius, Velebil (LMCS'11)
- Rational weighted languages for Noetherian semirings S — Bonsangue, Milius, Silva (TOCL'11)

Different Parameters for $\mathcal{C}$ and H

Instances of ϱH

- Rational Σ-trees for a signature Σ (on Set)
- Eventually periodic streams (on Set)
- Rational streams (on Vec_K) Milius (LICS'10)
- Rational λ-trees (on $\mathsf{Set}^{\mathscr{F}}$) Adámek, Milius, Velebil (LMCS'11)
- Rational weighted languages for Noetherian semirings S Bonsangue, Milius, Silva (TOCL'11)

Most likely not instances

- Courcelle's algebraic trees Adámek, Milius, Velebil (TCS'11)
- Context-free languages Winter, Bonsangue, Rutten (LMCS'13)

Different Parameters for $\mathcal{C}$ and H

Instances of ϱH

- Rational Σ-trees for a signature Σ (on Set)
- Eventually periodic streams (on Set)
- Rational streams (on Vec_K) — Milius (LICS'10)
- Rational λ-trees (on $\mathsf{Set}^{\mathscr{F}}$) — Adámek, Milius, Velebil (LMCS'11)
- Rational weighted languages for Noetherian semirings S — Bonsangue, Milius, Silva (TOCL'11)

Most likely not instances

- Courcelle's algebraic trees — Adámek, Milius, Velebil (TCS'11)
- Context-free languages — Winter, Bonsangue, Rutten (LMCS'13)

Problem

$\varrho H \to \nu H$ not always monic. Not all equal behaviours are identified!

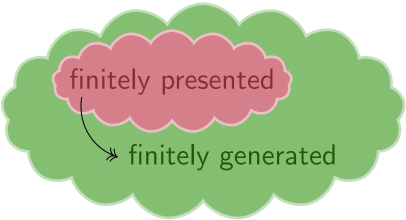
Problem & Task

Background: LFP Categories = Notions of finiteness

finitely presented

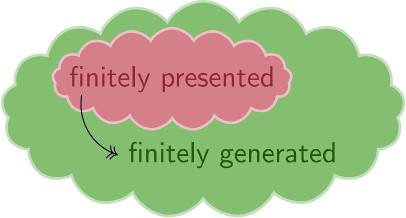
Problem & Task

Background: LFP Categories = Notions of finiteness



Problem & Task

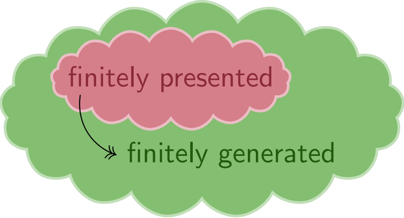
Background: LFP Categories = Notions of finiteness



- fp = fg: Set, Vec_K

Problem & Task

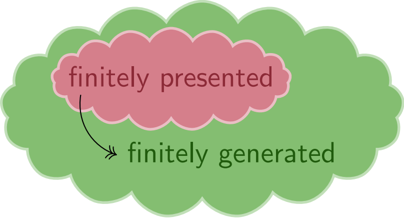
Background: LFP Categories = Notions of finiteness



- fp = fg: Set, Vec_K
- fp ≠ fg: Monoids, Groups, $\mathsf{Mnd}_{\mathsf{f}}(\mathsf{Set})$,

Problem & Task

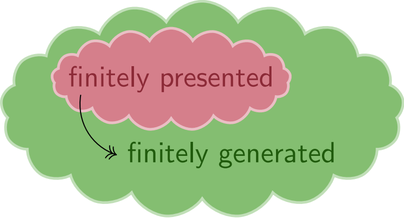
Background: LFP Categories = Notions of finiteness



- fp = fg: Set, Vec_K
- fp $\neq$ fg: Monoids, Groups, $\mathsf{Mnd}_f(\mathsf{Set})$,
- fp $\overset{?}{=}$ fg: Idempotent Semirings, ...

Problem & Task

Background: LFP Categories = Notions of finiteness



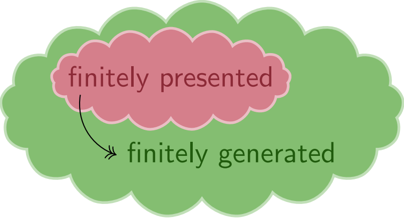
- fp = fg: Set, Vec_K
- fp $\neq$ fg: Monoids, Groups, $\mathsf{Mnd}_\mathsf{f}(\mathsf{Set})$,
- fp $\stackrel{?}{=}$ fg: Idempotent Semirings, ...

Wanted:

$$\begin{array}{c}\text{Collection of}\\\text{finite behaviours}\end{array} = \begin{array}{l}\text{Subcoalgebra of } \nu H\\\text{for finite behaviours}\end{array}$$

Problem & Task

Background: LFP Categories = Notions of finiteness



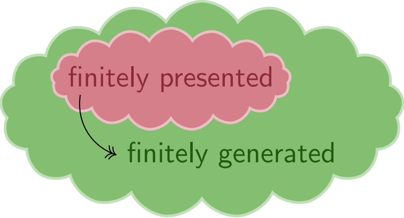
- fp = fg: Set, Vec_K
- fp ≠ fg: Monoids, Groups, $\mathrm{Mnd}_{\mathrm{f}}(\mathrm{Set})$,
- fp $\overset{?}{=}$ fg: Idempotent Semirings, ...

Wanted:

Collection of finite behaviours = Subcoalgebra of νH for finite behaviours $\neq \varrho H$

Problem & Task

Background: LFP Categories = Notions of finiteness



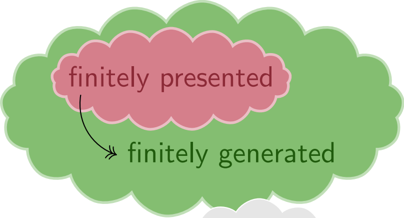
- fp = fg: Set, Vec_K
- fp ≠ fg: Monoids, Groups, $\mathrm{Mnd}_\mathrm{f}(\mathrm{Set})$,
- fp $\overset{?}{=}$ fg: Idempotent Semirings, . . .

Wanted:

$$\color{red}{???} = \begin{array}{c}\text{Collection of}\\\text{finite behaviours}\end{array} = \begin{array}{c}\text{Subcoalgebra of } \nu H\\\text{for finite behaviours}\end{array} \neq \varrho H$$

Problem & Task

Background: LFP Categories = Notions of finiteness



- fp = fg: Set, Vec_K
- fp $\neq$ fg: Monoids, Groups, $\mathrm{Mnd}_\mathrm{f}(\mathrm{Set})$,
- fp $\overset{?}{=}$ fg: Idempotent Semirings, …

Wanted:

$$??? = \text{Collection of finite behaviours} = \text{Subcoalgebra of } \nu H \text{ for finite behaviours} \neq \varrho H$$

Problem & Task

Background: LFP Categories = Notions of finiteness

finitely presented → finitely generated

- fp = fg: Set, Vec_K
- fp $\neq$ fg: Monoids, Groups, $\mathsf{Mnd}_{\mathsf{f}}(\mathsf{Set})$,
- fp $\overset{?}{=}$ fg: Idempotent Semirings, …

Wanted:

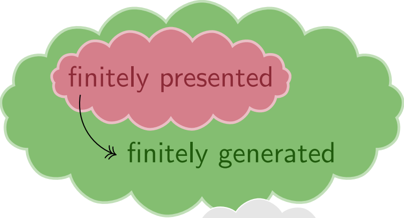
Universal property?

Fixpoint?

??? = Collection of finite behaviours = Subcoalgebra of νH for finite behaviours $\neq \varrho H$

Problem & Task

Background: LFP Categories = Notions of finiteness



- fp = fg: Set, Vec_K
- fp ≠ fg: Monoids, Groups, $\mathsf{Mnd}_\mathsf{f}(\mathsf{Set})$,
- fp $\stackrel{?}{=}$ fg: Idempotent Semirings, . . .

Meaning of inverse?

Wanted:

Universal property?

Fixpoint?

$$\textcolor{red}{???} = \begin{array}{c}\text{Collection of}\\ \text{finite behaviours}\end{array} = \begin{array}{c}\text{Subcoalgebra of } \nu H\\ \text{for finite behaviours}\end{array} \neq \varrho H$$

New Approach via Finitely Generated (f.g.) Objects

Coalgebras with f.g. carrier

$$C \longrightarrow HC$$

lfg coalgebras

fg

$$
\begin{tikzcd}
 & D \arrow[r] & HD \\
S \arrow[ur, tail, "\forall"] \arrow[r, tail, "\exists"] & P \arrow[u, tail] \arrow[r, "\exists"] & HP \arrow[u, tail]
\end{tikzcd}
$$

fg

New Approach via Finitely Generated (f.g.) Objects

Coalgebras with f.g. carrier

$$C \longrightarrow HC$$

lfg coalgebras

$$\begin{tikzcd} & & D \arrow[r] & HD \\ \text{fg} \arrow[r] & S \arrow[ur, tail, "\forall"] \arrow[r, tail, "\exists"] & P \arrow[u, tail] \arrow[r, "\exists"] & HP \arrow[u, tail] \\ & & \text{fg} \arrow[u] & \end{tikzcd}$$

Theorem: The **Locally Finite Fixpoint** of H

Suppose

- $\mathcal{C}$ LFP Category
- $H : \mathcal{C} \to \mathcal{C}$ mono-preserving and finitary endofunctor.

Final locally finitely generated (lfg) coalgebra ϑH exists and is

- subcoalgebra of νH.
- a fixpoint of H, i.e. has isomorphic structure

New Approach via Finitely Generated (f.g.) Objects

Coalgebras with f.g. carrier

$C \longrightarrow HC$

lfg coalgebras

$$\begin{tikzcd}
 & & D \arrow[r] & HD \\
\mathrm{fg} \arrow[r] & S \arrow[ur, tail, "\forall"] \arrow[r, tail, "\exists"] & P \arrow[u, tail] \arrow[r, "\exists"] & HP \arrow[u, tail] \\
 & & \mathrm{fg} \arrow[u] &
\end{tikzcd}$$

Theorem: The Locally Finite Fixpoint of H

Suppose

- $\mathcal{C}$ LFP Category
- $H : \mathcal{C} \to \mathcal{C}$ mono-preserving and finitary endofunctor.

Final locally finitely generated (lfg) coalgebra ϑH exists and is

- subcoalgebra of νH.
- a fixpoint of H, i.e. has isomorphic structure

The Inverse: Initial fg-iterative algebra

(A, α) fg-iterative $\Leftrightarrow$ solves $X \to HX + A$ (X fg) uniquely.

The Rational Image

Assume furthermore

- $\mathcal{C}$ has enough strong epi projectives
- H preserves strong epis

The Rational Image

Assume furthermore

- $\mathcal{C}$ has enough strong epi projectives
- H preserves strong epis

Examples

- Set, Vec_K
- Finitiary endofunctors $\mathsf{Fun}_f(\mathsf{Set})$
- Variety of algebras = Eilenberg-Moore category Set^T for T finitary monad, e.g. Groups, idempotent semi-rings, ...

The Rational Image

Assume furthermore

- $\mathcal{C}$ has enough strong epi projectives
- H preserves strong epis

Examples

- Set, Vec_K
- Finitiary endofunctors $\mathsf{Fun}_f(\mathsf{Set})$
- Variety of algebras = Eilenberg-Moore category Set^T for T finitary monad, e.g. Groups, idempotent semi-rings, ...

Theorem

$$\varrho H \twoheadrightarrow \vartheta H \rightarrowtail \nu H$$

The Rational Image

Assume furthermore

- $\mathcal{C}$ has enough strong epi projectives
- H preserves strong epis

Examples

- Set, Vec_K
- Finitiary endofunctors $\mathsf{Fun}_f(\mathsf{Set})$
- Variety of algebras = Eilenberg-Moore category Set^T for T finitary monad, e.g. Groups, idempotent semi-rings, ...

Theorem

$$\varrho H \twoheadrightarrow \vartheta H \rightarrowtail \nu H$$

Whenever fp=fg

$$\varrho H \xrightarrow{\cong} \vartheta H \rightarrowtail \nu H$$

Applications

Whenever fp=fg, e.g. in Set, Nominal Sets, and Vec_K

$$\varrho H = \vartheta H$$

Applications

Whenever fp=fg, e.g. in Set, Nominal Sets, and Vec_K

$$\varrho H = \vartheta H$$

Inherited examples

- Rational Σ-trees
- Eventually periodic streams
- Rational λ-trees modulo α-equivalence
- Rational weighted languages for Noetherian semirings S

$\vdots$

Applications

Whenever fp=fg, e.g. in Set, Nominal Sets, and Vec_K

$$\varrho H = \vartheta H$$

Inherited examples

- Rational Σ-trees
- Eventually periodic streams
- Rational λ-trees modulo α-equivalence
- Rational weighted languages for Noetherian semirings S

$\vdots$

Interesting Examples

$$\text{finitely presented} \subsetneq \text{finitely generated}$$

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in Set}}{x^\sharp : TX \to \bar{H}TX \text{ in Set}^T}$$

Computational side effect

its determinization

Silva, Bonchi, Bonsangue, Rutten'13

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in } \mathsf{Set}}{x^\sharp : TX \to \bar{H}TX \text{ in } \mathsf{Set}^T}$$

Computational side effect

its determinization

Silva, Bonchi, Bonsangue, Rutten'13

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "x"'] & TX \arrow[dl, "x^\sharp"] \\
HTX &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\nu H \arrow[d, "\tau", "\cong"'] \\
H\nu H
\end{tikzcd}$$

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in } \mathsf{Set}}{x^\sharp : TX \to \bar{H}TX \text{ in } \mathsf{Set}^T}$$

Computational side effect

its determinization

Silva, Bonchi, Bonsangue, Rutten'13

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, "x"'] & TX \arrow[rr, dashed, "x^{\sharp\dagger}"] \arrow[dl, "x^\sharp"] & & \nu H \arrow[d, "\tau", "\cong"'] \\
HTX \arrow[rrr, "Hx^{\sharp\dagger}"] & & & H\nu H
\end{tikzcd}$$

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in } \mathsf{Set}}{x^\sharp : TX \to \bar{H}TX \text{ in } \mathsf{Set}^T}$$

its determinization

Computational side effect

Silva, Bonchi, Bonsangue, Rutten'13

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, "x"'] & \mathcal{P}_\mathrm{f}X \arrow[rr, dashed, "x^{\sharp\dagger}"] \arrow[dl, "x^\sharp"] & & \nu H \arrow[d, "\tau", "\cong"'] \\
2 \times \mathcal{P}_\mathrm{f}X^\Sigma \arrow[rrr, "Hx^{\sharp\dagger}"] & & & H\nu H
\end{tikzcd}$$

Non-Determinism

$T = \mathcal{P}_\mathrm{f}$

$H = 2 \times (-)^\Sigma$

Set^T = Join Semi-Lattices

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in Set}}{x^\sharp : TX \to \bar{H}TX \text{ in Set}^T}$$

Computational side effect

its determinization

Silva, Bonchi, Bonsangue, Rutten'13

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, "x"'] & \mathcal{P}_f X \arrow[r, dashed, "x^{\sharp\dagger}"] \arrow[dl, "x^\sharp"] & \nu H \arrow[d, "\tau", "\cong"'] \\
2 \times \mathcal{P}_f X^\Sigma \arrow[rr, "Hx^{\sharp\dagger}"] & & H\nu H
\end{tikzcd}$$

Non-Determinism

$T = \mathcal{P}_f$

$H = 2 \times (-)^\Sigma$

Set^T = Join Semi-Lattices

Proposition

$$\vartheta \bar{H} = \bigcup_{\substack{x : X \to HTX \\ x \text{ finite}}} \mathrm{Im}(x^{\sharp\dagger}) \subseteq \nu H$$

Generalized Powerset Construction $\mathcal{C} = \mathsf{Set}^T$

T-Automaton

$$\frac{x : X \to HTX \text{ in Set}}{x^\sharp : TX \to \bar{H}TX \text{ in Set}^T}$$

its determinization

Computational side effect

Silva, Bonchi, Bonsangue, Rutten'13

$$\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, "x"'] & TX \arrow[r, dashed, "x^{\sharp\dagger}"] \arrow[dl, "x^\sharp"] & \vartheta\bar{H} \arrow[r, hook, "r^\dagger"] \arrow[d, "r"] & \nu H \arrow[d, "\tau", "\cong"'] \\
HTX \arrow[rr, "Hx^{\sharp\dagger}"] & & H\vartheta\bar{H} \arrow[r, "Hr^\dagger"] & H\nu H
\end{tikzcd}$$

Non-Determinism

$T = \mathcal{P}_\mathsf{f}$

$H = 2 \times (-)^\Sigma$

Set^T = Join Semi-Lattices

Proposition

$$\vartheta\bar{H} = \bigcup_{\substack{x : X \to HTX \\ x \text{ finite}}} \mathrm{Im}(x^{\sharp\dagger}) \subseteq \nu H$$

Stack Machines

Initial stack $\to$ output

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\stackrel{?}{=}$ fg

Stack configurations

Stack Machines

Initial stack $\to$ output

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\overset{?}{=}$ fg

Stack configurations

Finite state HT-Coalgebras

real-time context-free languages

Goncharov, Milius, Silva'14

Stack Machines

Initial stack $\to$ output

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\overset{?}{=}$ fg

Stack configurations

Finite state HT-Coalgebras

real-time context-free languages Goncharov, Milius, Silva'14

$\vartheta \bar{H} \approx$ real-time cf. languages

Stack Machines

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

Initial stack $\to$ output

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\stackrel{?}{=}$ fg

Stack configurations

Finite state HT-Coalgebras

real-time context-free languages

Goncharov, Milius, Silva'14

$\vartheta \bar{H} \approx$ real-time cf. languages

$T' = T \otimes$ non-determinism

non-deterministic stack monad

Goncharov, Milius, Silva'14

EM-Algebras

fp $\stackrel{?}{=}$ fg

Non-Det. Stack configurations

Stack Machines

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

Initial stack $\to$ output

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\overset{?}{=}$ fg

Stack configurations

Finite state HT-Coalgebras

real-time context-free languages

Goncharov, Milius, Silva'14

$\vartheta \bar{H} \approx$ real-time cf. languages

$T' = T \otimes$ non-determinism

non-deterministic stack monad

Goncharov, Milius, Silva'14

EM-Algebras

fp $\overset{?}{=}$ fg

Non-Det. Stack configurations

Finite state HT'-Coalgebras

context-free languages

Goncharov, Milius, Silva'14

Stack Machines

$$HX = B \times X^{\Sigma} \quad B \subseteq 2^{\Gamma^*}$$

Initial stack $\to$ output

T = Stack monad

Submonad of the store monad $(- \times \Gamma^*)^{\Gamma^*}$. Goncharov'13

EM-Algebras

fp $\overset{?}{=}$ fg

Stack configurations

Finite state HT-Coalgebras

real-time context-free languages

Goncharov, Milius, Silva'14

$\vartheta \bar{H} \approx$ real-time cf. languages

$T' = T \otimes$ non-determinism

non-deterministic stack monad

Goncharov, Milius, Silva'14

EM-Algebras

fp $\overset{?}{=}$ fg

Non-Det. Stack configurations

Finite state HT'-Coalgebras

context-free languages

Goncharov, Milius, Silva'14

$\vartheta \tilde{H} \approx$ context-free languages

S-Algebraic Power Series and $HX = S \times X^{\Sigma}$

S commutative semiring

Formal Power Series $X^* \to S$

$S\langle X\rangle$ = Polynomials

S-Algebraic Power Series and $HX = S \times X^{\Sigma}$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X\rangle$ = Polynomials

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

S-Algebraic Power Series and $HX = S \times X^{\Sigma}$

S commutative semiring

Formal Power Series $X^* \to S$

$S\langle X\rangle$ = Polynomials

fp $\overset{?}{\neq}$ fg

Monad $T = S\langle - + \Sigma\rangle$

EM-Algebras = S-algebras with Σ-pointing

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

S-Algebraic Power Series and $HX = S \times X^{\Sigma}$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X\rangle$ = Polynomials

Monad $T = S\langle - + \Sigma\rangle$ fp $\overset{?}{=}$ fg

EM-Algebras = S-algebras with Σ-pointing

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

Monad $T = \mathcal{P}_{\mathsf{f}}((- + \Sigma)^*)$

EM-Algebras = Idempotent semirings with Σ-pointing

S-Algebraic Power Series and $HX = S \times X^\Sigma$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X\rangle$ = Polynomials

fp $\overset{?}{=}$ fg

Monad $T = S\langle - + \Sigma\rangle$

EM-Algebras = S-algebras with Σ-pointing

Finite state $x : X \to HS\langle X\rangle$

Weighted CF Grammars
Winter, Bonsangue, Rutten'15

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

Monad $T = \mathcal{P}_f((- + \Sigma)^*)$

EM-Algebras = Idempotent semirings with Σ-pointing

S-Algebraic Power Series and $HX = S \times X^\Sigma$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X\rangle$ = Polynomials

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

fp $\overset{?}{=}$ fg

Monad $T = S\langle - + \Sigma\rangle$

EM-Algebras = S-algebras with Σ-pointing

Monad $T = \mathcal{P}_f((- + \Sigma)^*)$

EM-Algebras = Idempotent semirings with Σ-pointing

Finite state $x : X \to HS\langle X\rangle$

Weighted CF Grammars
Winter, Bonsangue, Rutten'15

Finite state $x : X \to HTX$

CF Grammars in weak Greibach Normal Form
Winter, Bonsangue, Rutten'13

S-Algebraic Power Series and $HX = S \times X^{\Sigma}$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X \rangle$ = Polynomials

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

Monad $T = S\langle - + \Sigma \rangle$ fp $\overset{?}{=}$ fg

EM-Algebras = S-algebras with Σ-pointing

Monad $T = \mathcal{P}_f((- + \Sigma)^*)$
EM-Algebras = Idempotent semirings with Σ-pointing

Finite state $x : X \to HS\langle X \rangle$

Weighted CF Grammars
Winter, Bonsangue, Rutten'15

Finite state $x : X \to HTX$
CF Grammars in weak Greibach Normal Form
Winter, Bonsangue, Rutten'13

$\vartheta\bar{H}$ in Set^T

Constructively S-algebraic FPS.

S-Algebraic Power Series and $HX = S \times X^\Sigma$

S commutative semiring

Formal Power Series $X^* \to S$
$S\langle X\rangle$ = Polynomials

$S = 2$ = Boolean Semiring

Formal Languages $X^* \to 2$

fp $\overset{?}{=}$ fg

Monad $T = S\langle - + \Sigma\rangle$

EM-Algebras = S-algebras with Σ-pointing

Monad $T = \mathcal{P}_f((- + \Sigma)^*)$

EM-Algebras = Idempotent semirings with Σ-pointing

Finite state $x : X \to HS\langle X\rangle$

Weighted CF Grammars
Winter, Bonsangue, Rutten'15

Finite state $x : X \to HTX$

CF Grammars in weak Greibach Normal Form
Winter, Bonsangue, Rutten'13

$\vartheta\bar{H}$ in Set^T

Constructively S-algebraic FPS.

$\vartheta\bar{H}$ in Set^T

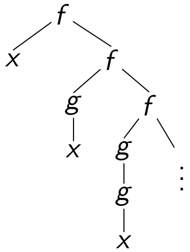
Context-free Σ-Languages

Algebraic Trees & Recursive Program Schemes

Example: For the signature $\Sigma = \{{}^{f}\!/_{2}, {}^{g}\!/_{1}\}$

$$\varphi(x) = f(x, \varphi(g(x)))$$

Solution of $\varphi(x)$



Algebraic Trees & Recursive Program Schemes

Example: For the signature $\Sigma = \{{}^{f}/_{2}, {}^{g}/_{1}\}$

$$\varphi(x) = f(x, \varphi(g(x)))$$

Coalgebraically

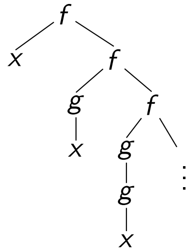
fp≠fg

$\mathcal{C}$ = finitary H_Σ-pointed Set-Monads:

RPS $\longrightarrow$ $e : B \longrightarrow H_\Sigma \cdot B + \mathsf{Id}$

Adámek, Milius, Velebil'11

Solution of $\varphi(x)$



Algebraic Trees & Recursive Program Schemes

Example: For the signature $\Sigma = \{{}^{f}/_{2}, {}^{g}/_{1}\}$

$$\varphi(x) = f(x, \varphi(g(x)))$$

Coalgebraically

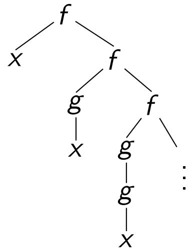
fp≠fg

$\mathcal{C}$ = finitary H_Σ-pointed Set-Monads:

$$\text{RPS} \longrightarrow e : B \longrightarrow H_\Sigma \cdot B + \mathsf{Id}$$

Adámek, Milius, Velebil'11

Solution of $\varphi(x)$



Proposition

Monad of algebraic trees $= \vartheta(H_\Sigma \cdot (-) + \mathsf{Id})$

Algebraic Trees & Recursive Program Schemes

Example: For the signature $\Sigma = \{f/2, g/1\}$

$$\varphi(x) = f(x, \varphi(g(x)))$$

Coalgebraically

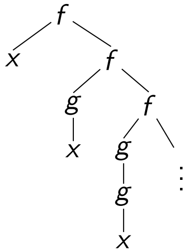
fp≠fg

$\mathcal{C}$ = finitary H_Σ-pointed Set-Monads:

$$\text{RPS} \longrightarrow e : B \longrightarrow H_\Sigma \cdot B + \mathrm{Id}$$

Adámek, Milius, Velebil'11

Solution of $\varphi(x)$



Proposition

Monad of algebraic trees $= \vartheta(H_\Sigma \cdot (-) + \mathrm{Id})$

Solves open question about its Universal Property

Conclusions

The locally finite fixpoint …

- one abstract framework

Conclusions

The locally finite fixpoint . . .

- one abstract framework
- characterized by two universal properties

Conclusions

The locally finite fixpoint ...

- one abstract framework
- characterized by two universal properties
- many different instances

Conclusions

The locally finite fixpoint ...

- one abstract framework
- characterized by two universal properties
- many different instances

Future Work

- Further instances? Tape Machines?
- General syntactic descriptions of the LFF?

[AMV06] Jiří Adámek, Stefan Milius, Jiří Velebil. "Iterative Algebras at Work". In: **Mathematical Structures in Computer Science** 16.6 (2006), pp. 1085–1131. DOI: 10.1017/S0960129506005706.

[AMV11a] Jiří Adámek, Stefan Milius, Jiří Velebil. "On second-order iterative monads". In: **Theoretical Computer Science** 412.38 (2011). {CMCS} Tenth Anniversary Meeting, pp. 4969–4988. ISSN: 0304-3975. DOI: http://dx.doi.org/10.1016/j.tcs.2011.04.027. URL: http://www.sciencedirect.com/science/article/pii/S0304397511003252.

[AMV11b] Jiří Adámek, Stefan Milius, Jiří Velebil. "Semantics of higher-order recursion schemes". In: **Log. Methods Comput. Sci.** 7.1:15 (2011), 43 pp.

[BMS13] Marcello Bonsangue, Stefan Milius, Alexandra Silva. "Sound and complete axiomatizations of coalgebraic language equivalence". In: **ACM Trans. Comput. Log.** 14.1:7 (2013), 52 pp.

[GMS14] Sergey Goncharov, Stefan Milius, Alexandra Silva. "Towards a Coalgebraic Chomsky Hierarchy". In: **Proc. 8th IFIP TC 1/WG 2.2 International Conference on Theoretical Computer Science (TCS'14)**. Ed. by Josep Diaz, Ivan Lanese, Davide Sangiorgi. Vol. 8705. Lecture Notes Comput. Sci. Springer, 2014, pp. 265–280.

[Gon13] Sergey Goncharov. "Trace Semantics via Generic Observations". In: **Algebra and Coalgebra in Computer Science (CALCO 2013)**. Ed. by Reiko Heckel, Stefan Milius. Lecture Notes in Computer Science. Springer, 2013.

[Mil10] Stefan Milius. “A Sound and Complete Calculus for finite Stream Circuits”. In: **Proc. 25th Annual Symposium on Logic in Computer Science (LICS’10)**. 2010, pp. 449–458.

[SBBR09] Alexandra Silva, Filippo Bonchi, Marcello M. Bonsangue, Jan J. M. M. Rutten. “Deriving Syntax and Axioms for Quantitative Regular Behaviours”. In: **CONCUR 2009 - Concurrency Theory, 20th International Conference, CONCUR 2009, Bologna, Italy, September 1-4, 2009. Proceedings**. 2009, pp. 146–162. DOI: 10.1007/978-3-642-04081-8_11. URL: http://dx.doi.org/10.1007/978-3-642-04081-8_11.

[SBBR13] Alexandra Silva, Filippo Bonchi, Marcello M. Bonsangue, Jan J. M. M. Rutten. "Generalizing determinization from automata to coalgebras". In: **Logical Methods in Computer Science** 9.1 (2013). DOI: 10.2168/LMCS-9(1:9)2013. URL: http://dx.doi.org/10.2168/LMCS-9(1:9)2013.

[WBR13] Joost Winter, Marcello Bonsangue, Jan Rutten. "Coalgebraic Characterizations of Context-Free Languages". en. In: **Logical Methods in Computer Science** 9.3 (Sept. 2013). ISSN: 1860-6974. URL: http://arxiv.org/pdf/1308.1228.pdf.

[WBR15] Joost Winter, Marcello M. Bonsangue, Jan J.M.M. Rutten. "Context-free coalgebras". In: **J. Comput. System Sci.** 81.5 (2015), pp. 911–939.