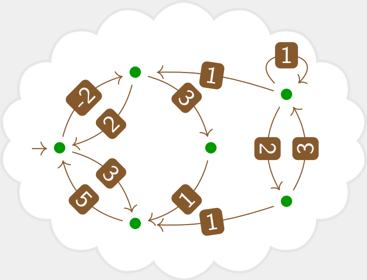
Minimality Notions via Factorization Systems

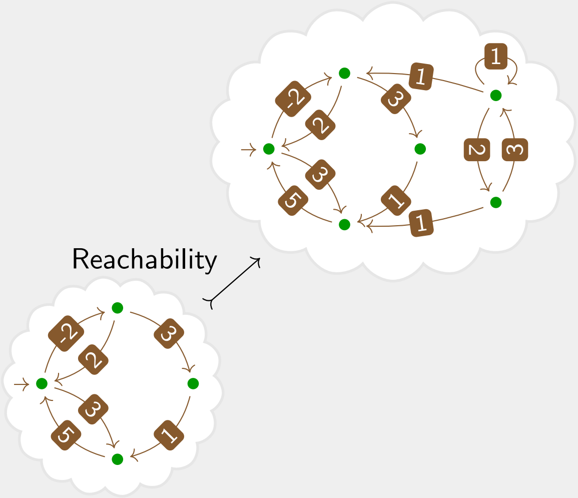
(Calco Pearl)

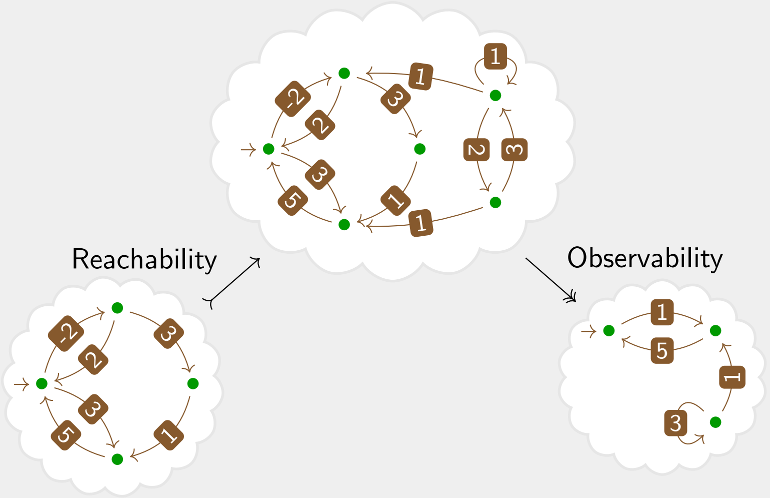
Thorsten Wißmann

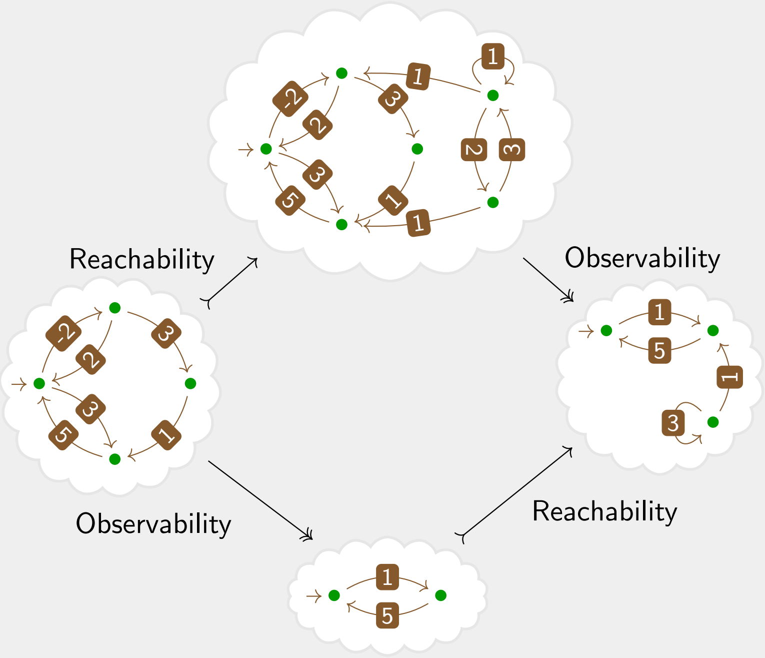
Radboud University, Nijmegen, the Netherlands

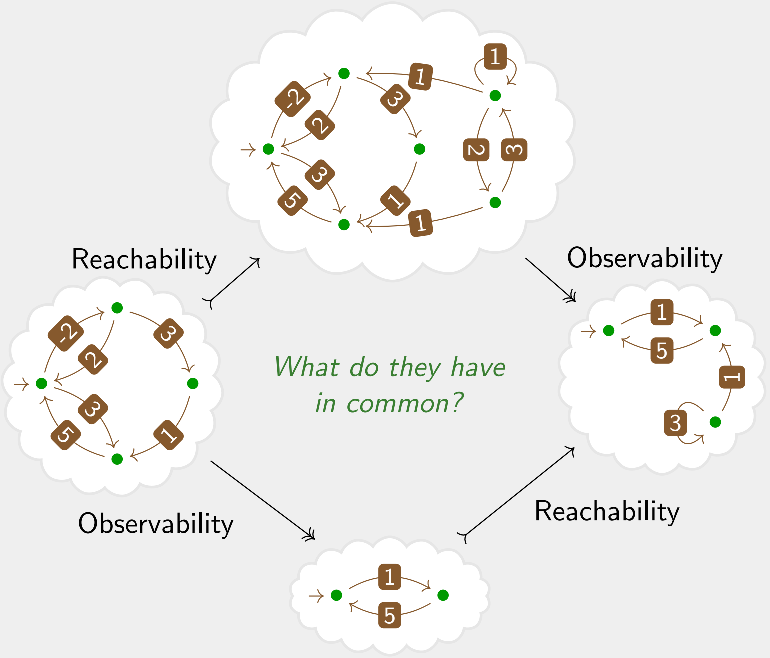
Calco 2021
September 2, 2021


1
1
3
-2
2
2
3
3
5
1
1


Reachability
-2
2
3
1
1
2
3
3
5
1
1
-2
2
3
3
5
1


Reachability
Observability


Reachability
Observability
Observability
Reachability


Reachability
Observability
What do they have in common?
Observability
Reachability

States Successor map Successor type

Category Coalg(F)

For a functor $F\colon \mathcal{C} \to \mathcal{C}$: ($F$-)coalgebra $C \xrightarrow{c} FC$

Coalgebra homomorphisms:

$h\colon (C,c) \longrightarrow (D,d)$ in Coalg(F) $:\Longleftrightarrow$

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "h"'] & FC \arrow[d, "Fh"] \\ D \arrow[r, "d"] & FD \end{tikzcd}$$

States Successor map Successor type

Category Coalg(F)

For a functor $F\colon \mathcal{C} \to \mathcal{C}$: (F-)coalgebra $C \xrightarrow{c} FC$

Coalgebra homomorphisms:

$h\colon (C, c) \longrightarrow (D, d)$ in Coalg(F) $:\Longleftrightarrow$

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "h"'] & FC \arrow[d, "Fh"] \\ D \arrow[r, "d"] & FD \end{tikzcd}$$

Examples

From now on: $\mathcal{C} = \mathsf{Set}$. Coalgebras for functors $F\colon \mathsf{Set} \to \mathsf{Set}$:

- $FX = 2 \times X^A$ deterministic automata (without initial state)
- $FX = \mathcal{P}X$ transition systems
- $FX = \mathbb{R}^{(X)}$ weighted systems (negative weights!)

Coalgebraic Task

$$\begin{tikzcd} \text{Given a coalgebra} & C \arrow[r, "c"] \arrow[d, two heads, "h"'] & FC \arrow[d, "Fh"] \\ \text{find the simple quotient} & D \arrow[r, "d"] & FD \end{tikzcd}$$

no proper quotient

Gumm '03

Coalgebraic Task

$$\text{Given a coalgebra} \quad \begin{tikzcd} C \arrow[r, "c"] \arrow[d, two heads, "h"'] & FC \arrow[d, "Fh"] \\ D \arrow[r, "d"] & FD \end{tikzcd}$$
$$\text{find the simple quotient}$$

no proper quotient

Gumm '03

Examples

This task for specific F-coalgebras:

- $FX = 2 \times X^A \rightsquigarrow$ quotient by: language equivalence
- $FX = \mathcal{P}X \rightsquigarrow$ quotient by: (strong) bisimilarity
- $FX = \mathbb{R}^{(X)} \rightsquigarrow$ quotient by: weighted bisimilarity

Category $\mathsf{Coalg}_I(F)$ for $I \in \mathsf{Set}$

- I-pointed F-coalgebra (C, c, i_C) $\quad I \xrightarrow{i_C} C \xrightarrow{c} FC$
- Homomorphism:

$$
\begin{tikzcd}
(C, c, i_C) \arrow[d, "h"] \\
(D, d, i_D)
\end{tikzcd}
\text{ in } \mathsf{Coalg}_I(F) \iff
\begin{tikzcd}
I \arrow[r, "i_C"] \arrow[dr, "i_D"'] & C \arrow[r, "c"] \arrow[d, "h"'] & FC \arrow[d, "Fh"] \\
& D \arrow[r, "d"] & FD
\end{tikzcd}
\text{ in } \mathcal{C}
$$

Category $\mathsf{Coalg}_I(F)$ for $I \in \mathsf{Set}$

- I-pointed F-coalgebra (C, c, i_C) $\quad I \xrightarrow{i_C} C \xrightarrow{c} FC$
- Homomorphism:

$$
\begin{array}{c} (C,c,i_C) \\ \big\downarrow h \\ (D,d,i_D) \end{array}
\quad \text{in } \mathsf{Coalg}_I(F) \iff
\begin{tikzcd}
I \arrow[r, "i_C"] \arrow[dr, "i_D"'] & C \arrow[r, "c"] \arrow[d, "h"'] & FC \arrow[d, "Fh"] \\
& D \arrow[r, "d"] & FD
\end{tikzcd}
\quad \text{in } \mathcal{C}
$$

Examples

Coalgebras for Set-functors $F\colon \mathsf{Set} \to \mathsf{Set}$ and $I = 1 = \{*\}$:

- $FX = 2 \times X^A$ deterministic automata incl. initial state
- $FX = \mathcal{P}X$ pointed graphs / transition systems
- $FX = \mathbb{R}^{(X)}$ pointed weighted systems

Coalgebraic Task

$$\begin{tikzcd} \text{Given a pointed coalgebra} & I \arrow[r, "i_C"] \arrow[dr, "i_D"'] & C \arrow[r, "c"] & FC \\ \text{find the reachable subcoalgebra} & & D \arrow[u, tail, "h"] \arrow[r, "d"] & FD \arrow[u, "Fh"'] \end{tikzcd}$$

no proper
pointed subcoalgebra

Adámek, Milius, Moss, Sousa '13

Coalgebraic Task

Given a pointed coalgebra

find the reachable subcoalgebra

$$\begin{tikzcd} I \arrow[r, "i_C"] \arrow[dr, "i_D"'] & C \arrow[r, "c"] & FC \\ & D \arrow[u, hook, "h"] \arrow[r, "d"] & FD \arrow[u, "Fh"'] \end{tikzcd}$$

no proper
pointed subcoalgebra

Adámek, Milius, Moss, Sousa '13

Examples

This task for specific F-coalgebras:

- $FX = 2 \times X^A \rightsquigarrow$ restrict to: reachable states
- $FX = \mathcal{P}X \rightsquigarrow$ restrict to: reachable states
- $FX = \mathbb{R}^{(X)} \rightsquigarrow$ restrict to: reachable states

Definition: $\mathcal{M}$-Minimality

Category $\mathcal{K}$ with an $(\mathcal{E},\mathcal{M})$-factorization system:

in $\mathcal{M}$

object C is ($\mathcal{M}$-)*minimal* $:\Longleftrightarrow$ every $h\colon D \rightarrowtail C$ is an isomorphism

Definition: $\mathcal{M}$-Minimality

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

in $\mathcal{M}$

object C is $(\mathcal{M}\text{-})$*minimal* $:\Longleftrightarrow$ every $h\colon D \rightarrowtail C$ is an isomorphism

$\mathcal{K} = \mathsf{Coalg}_I(F)$

(C, c, i_C) is ($\mathcal{M}$-carried-)minimal

$\Longleftrightarrow$

(C, c, i_C) is reachable

Definition: $\mathcal{M}$-Minimality

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

in $\mathcal{M}$

object C is $(\mathcal{M}\text{-})$*minimal* $:\Longleftrightarrow$ every $h\colon D \rightarrowtail C$ is an isomorphism

$\mathcal{K} = \mathrm{Coalg}_I(F)$

(C, c, i_C) is ($\mathcal{M}$-carried-)minimal

$\Longleftrightarrow$

(C, c, i_C) is reachable

$\mathcal{K} = \mathrm{Coalg}(F)^{\mathrm{op}}$

(C, c) is ($\mathcal{E}$-carried-)minimal

$\Longleftrightarrow$

(C, c) is simple

Definition: $(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{K}$

Classes of Morphisms: $\rightarrowtail$ $\mathcal{M}$ $\twoheadrightarrow$ $\mathcal{E}$

each closed under composition with isomorphisms

Definition: $(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{K}$

Classes of Morphisms: $\rightarrowtail$ $\mathcal{M}$ $\twoheadrightarrow$ $\mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f\colon A \to B$

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, tail, "m"] & B \end{tikzcd}$$

Definition: $(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{K}$

Classes of Morphisms: $\rightarrowtail$ $\mathcal{M}$ $\twoheadrightarrow$ $\mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f\colon A \to B$

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, tail, "m"] & B \end{tikzcd}$$

- Diagonal fill-in:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists! d"'] \\ C \arrow[r, tail, "m"] & D \end{tikzcd}$$

Definition: $(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{K}$

Classes of Morphisms: $\rightarrowtail$ $\mathcal{M}$ $\twoheadrightarrow$ $\mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f\colon A \to B$

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, tail, "m"] & B \end{tikzcd}$$

- Diagonal fill-in:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists! d"'] \\ C \arrow[r, tail, "m"] & D \end{tikzcd}$$

Example:

$\mathcal{K} = \mathrm{Set}$ $\quad$ $\mathcal{E} =$ surjective maps $\quad$ $\mathcal{M} =$ injective maps

Definition: $(\mathcal{E},\mathcal{M})$-factorization system in $\mathcal{K}$

Classes of Morphisms: $\rightarrowtail$ $\mathcal{M}$ $\twoheadrightarrow$ $\mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f\colon A \to B$

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, tail, "m"] & B \end{tikzcd}$$

- Diagonal fill-in:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists! d"'] \\ C \arrow[r, tail, "m"] & D \end{tikzcd}$$

Example:

$\mathcal{K} = \mathsf{Set}$ $\quad$ $\mathcal{E}$ = surjective maps $\quad$ $\mathcal{M}$ = injective maps

Example:

$(\mathcal{E},\mathcal{M})$-factorization system in $\mathcal{K}$ $\iff$ $(\mathcal{M},\mathcal{E})$-factorization system in $\mathcal{K}^{\mathbf{op}}$

Theorem

$(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{C}$ lifts to
an ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in:

- Coalg(F) if $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{M}$

Theorem

$(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{C}$ lifts to
an ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in:

- $\mathsf{Coalg}(F)$ if $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{M}$
- $\mathsf{Alg}(F)$ if $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$
- Eilenberg-Moore category of T if $T\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$

Theorem

$(\mathcal{E}, \mathcal{M})$-factorization system in $\mathcal{C}$ lifts to
an ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in:

- $\mathrm{Coalg}(F)$ if $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{M}$
- $\mathrm{Alg}(F)$ if $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$
- Eilenberg-Moore category of T if $T\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$

If $\mathcal{C} = \mathrm{Set}$, $(\mathcal{E}, \mathcal{M}) = (\mathrm{Epi}, \mathrm{Mono})$

- mono-preservation of F is wlog for coalgebras
- all Set-functors preserve epimorphisms

Proposition:

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

$$\text{object } C \text{ is } \mathcal{M}\text{-minimal} \quad \Longleftrightarrow \quad \text{every } h\colon D \to C \text{ is in } \mathcal{E}$$

Proposition:

Category $\mathcal{K}$ with an $(\mathcal{E},\mathcal{M})$-factorization system:

$$\text{object } C \text{ is } \mathcal{M}\text{-minimal} \iff \text{every } h\colon D \to C \text{ is in } \mathcal{E}$$

$\mathcal{K} = \mathsf{Coalg}_I(F)$

(C, c, i_C) is reachable

$\iff$

every $h\colon (D, d, i_D) \to (C, c, i_C)$ is surjective

Proposition:

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

$$\text{object } C \text{ is } \mathcal{M}\text{-minimal} \iff \text{every } h\colon D \to C \text{ is in } \mathcal{E}$$

$\mathcal{K} = \mathsf{Coalg}_I(F)$

(C, c, i_C) is reachable

$\iff$

every $h\colon (D, d, i_D) \to (C, c, i_C)$ is surjective

$\mathcal{K} = \mathsf{Coalg}(F)^{\mathsf{op}}$

(C, c) is simple

$\iff$

every $h\colon (C, c) \to (D, d)$ is injective

Minimality Notions

$\mathcal{M}$-minimality of X	$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$

Minimality Notions

$\mathcal{M}$-minimality of X	$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$
if $\mathcal{E} =$ Epi & $\mathcal{K}$ has equalizers: $\Leftrightarrow$ all parallel $X \rightrightarrows Y$ equal	if $\mathcal{M} =$ Mono & $\mathcal{K}$ has coequalizers: $\Leftrightarrow$ all parallel $Y \rightrightarrows X$ equal $\Leftrightarrow$ 'X subterminal'

Minimality Notions

$\mathcal{M}$-minimality of X	$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$
if $\mathcal{E} =$ Epi & $\mathcal{K}$ has equalizers: $\Leftrightarrow$ all parallel $X \rightrightarrows Y$ equal	if $\mathcal{M} =$ Mono & $\mathcal{K}$ has coequalizers: $\Leftrightarrow$ all parallel $Y \rightrightarrows X$ equal $\Leftrightarrow$ 'X subterminal'

Coalg(F)

If $F\colon \mathcal{C} \to \mathcal{C}$ preserves weak kernel pairs..

- monomorphisms in Coalg(F) = Mono-carried homomorphisms
- simple coalgebras = subterminal coalgebras

Definition:

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

$$\mathcal{M}\textit{-minimization} \text{ of } C \quad = \quad \begin{array}{c} m\colon D \rightarrowtail C \\ \text{where } D \text{ is } \mathcal{M}\text{-minimal} \end{array}$$

Definition:

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

$$\mathcal{M}\textit{-minimization} \text{ of } C \quad = \quad \begin{array}{c} m\colon D \rightarrowtail C \\ \text{where } D \text{ is } \mathcal{M}\text{-minimal} \end{array}$$

$\mathcal{K} = \mathrm{Coalg}_I(F)$

$\mathcal{M}$-minimization of (C, c, i_C)

$=$

reachable subcoalgebra of (C, c, i_C)

Definition:

Category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system:

$\mathcal{M}$*-minimization* of C $=$ $m\colon D \rightarrowtail C$ where D is $\mathcal{M}$-minimal

$\mathcal{K} = \mathsf{Coalg}_I(F)$

$\mathcal{M}$-minimization of (C, c, i_C)

$=$

reachable subcoalgebra of (C, c, i_C)

$\mathcal{K} = \mathsf{Coalg}(F)^{\mathsf{op}}$

$\mathcal{E}$-minimization of (C, c)

$=$

simple quotient of (C, c)

Proposition: $\mathcal{M}$-minimizations . . .

- are unique (up to iso) if $\mathcal{K}$ has pullbacks of $\mathcal{M}$-morphisms
- exist if $\mathcal{K}$ has wide pullbacks of $\mathcal{M}$-morphisms, $\mathcal{M} \subseteq$ Mono, and $\mathcal{K}$ is $\mathcal{M}$-wellpowered

Proposition: $\mathcal{M}$-minimizations ...

- are unique (up to iso) if $\mathcal{K}$ has pullbacks of $\mathcal{M}$-morphisms
- exist if $\mathcal{K}$ has wide pullbacks of $\mathcal{M}$-morphisms, $\mathcal{M} \subseteq$ Mono, and $\mathcal{K}$ is $\mathcal{M}$-wellpowered

$\mathcal{K} = \mathsf{Coalg}_I(F)$ $F\colon \mathsf{Set} \circlearrowleft$

reachable subcoalgebras ...

- are unique (up to iso) if F preserves finite intersections
- exist if F preserves infinite intersections

Adámek, Milius, Moss, Sousa '13

Proposition: $\mathcal{M}$-minimizations ...

- are unique (up to iso) if $\mathcal{K}$ has pullbacks of $\mathcal{M}$-morphisms
- exist if $\mathcal{K}$ has wide pullbacks of $\mathcal{M}$-morphisms, $\mathcal{M} \subseteq$ Mono, and $\mathcal{K}$ is $\mathcal{M}$-wellpowered

$\mathcal{K} = \mathrm{Coalg}_I(F)$ $\quad F\colon \mathrm{Set} \to \mathrm{Set}$

reachable subcoalgebras ...

- are unique (up to iso) if F preserves finite intersections
- exist if F preserves infinite intersections

Adámek, Milius, Moss, Sousa '13

$\mathcal{K} = \mathrm{Coalg}(F)^{\mathrm{op}}$ $\quad F\colon \mathrm{Set} \to \mathrm{Set}$

simple quotients ...

- are unique (up to iso)
- exist

Gumm '08

Assume: $\mathcal{M}$ is a class of monos and all $\mathcal{M}$-minimizations exist

Proposition:

If pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$, then

- $\mathcal{M}$-minimal objects form a coreflective subcategory $\mathcal{K}_{\mathsf{min}} \hookrightarrow \mathcal{K}$
- $\mathcal{M}$-minimization is the coreflector $\mathcal{K} \to \mathcal{K}_{\mathsf{min}}$ ($\Rightarrow$ functorial)
- $\mathcal{M}$-minimal objects are closed under $\mathcal{E}$-morphisms

Assume: $\mathcal{M}$ is a class of monos and all $\mathcal{M}$-minimizations exist

Proposition:

If pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$, then

- $\mathcal{M}$-minimal objects form a coreflective subcategory $\mathcal{K}_{\mathsf{min}} \hookrightarrow \mathcal{K}$
- $\mathcal{M}$-minimization is the coreflector $\mathcal{K} \to \mathcal{K}_{\mathsf{min}}$ ($\Rightarrow$ functorial)
- $\mathcal{M}$-minimal objects are closed under $\mathcal{E}$-morphisms

$\mathcal{K} = \mathsf{Coalg}_I(F)$ $\qquad$ $F\colon \mathsf{Set} \circlearrowleft$

If F preserves inverse images,

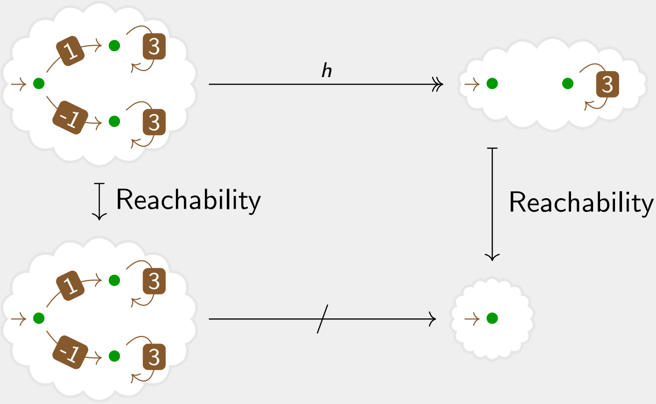
- coreflective subcategory $\mathsf{Coalg}_I^{\mathsf{reach}}(F) \hookrightarrow \mathsf{Coalg}_I(F)$
- reachability is functorial
- reachability closed under quotients

Inverse image preservation

- $FX = 2 \times X^A$ preserves inverse images
- $FX = \mathcal{P}X$ preserves inverse images
- $FX = \mathbb{R}^{(X)}$ does not preserve inverse images, because the monoid $(\mathbb{R}, +, 0)$ is not positive.

Inverse image preservation

- $FX = 2 \times X^A$ preserves inverse images
- $FX = \mathcal{P}X$ preserves inverse images
- $FX = \mathbb{R}^{(X)}$ does not preserve inverse images, because the monoid $(\mathbb{R}, +, 0)$ is not positive.



Assume: $\mathcal{M}$ is a class of monos and all $\mathcal{M}$-minimizations exist

Proposition:

If pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$, then

- $\mathcal{M}$-minimal objects form a coreflective subcategory $\mathcal{K}_{\mathsf{min}} \hookrightarrow \mathcal{K}$
- $\mathcal{M}$-minimization is the coreflector $\mathcal{K} \to \mathcal{K}_{\mathsf{min}}$ ($\Rightarrow$ functorial)
- $\mathcal{M}$-minimal objects are closed under $\mathcal{E}$-morphisms

$\mathcal{K} = \mathsf{Coalg}_I(F)$ $\qquad$ $F\colon \mathsf{Set} \circlearrowleft$

If F preserves inverse images,

- coreflective subcategory $\mathsf{Coalg}_I^{\mathsf{reach}}(F) \hookrightarrow \mathsf{Coalg}_I(F)$
- reachability is functorial
- reachability closed under quotients

Assume: $\mathcal{M}$ is a class of monos and all $\mathcal{M}$-minimizations exist

Proposition:

If pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$, then

- $\mathcal{M}$-minimal objects form a coreflective subcategory $\mathcal{K}_{\mathsf{min}} \hookrightarrow \mathcal{K}$
- $\mathcal{M}$-minimization is the coreflector $\mathcal{K} \to \mathcal{K}_{\mathsf{min}}$ ($\Rightarrow$ functorial)
- $\mathcal{M}$-minimal objects are closed under $\mathcal{E}$-morphisms

$\mathcal{K} = \mathsf{Coalg}_I(F)$ $\quad$ $F\colon \mathsf{Set} \circlearrowleft$

If F preserves inverse images,

- coreflective subcategory $\mathsf{Coalg}_I^{\mathsf{reach}}(F) \hookrightarrow \mathsf{Coalg}_I(F)$
- reachability is functorial
- reachability closed under quotients

$\mathcal{K} = \mathsf{Coalg}(F)^{\mathsf{op}}$ $\quad$ $F\colon \mathsf{Set} \circlearrowleft$

- Reflective subcategory $\mathsf{Coalg}^{\mathsf{simple}}(F) \hookrightarrow \mathsf{Coalg}(F)$
- Finding the simple quotient is functorial
- Simple closed under subcoalgebras

Gumm '08

Assume:

- all $\mathcal{M}$-minimizations in $\mathcal{K}$ and all $\mathcal{E}$-minimizations in $\mathcal{K}^{\mathbf{op}}$ exist
- $\mathcal{K}$ has pullbacks along $\mathcal{M}$
- $\mathcal{K}$ has pushouts along $\mathcal{E}$

Proposition

For $C \in \mathcal{K}$, the following yield the same:

- $\mathcal{E}$-minimization (in $\mathcal{K}^{\mathbf{op}}$) of the $\mathcal{M}$-minimization of C (in $\mathcal{K}$)
- $\mathcal{M}$-minimization (in $\mathcal{K}$) of the $\mathcal{E}$-minimization of C (in $\mathcal{K}^{\mathbf{op}}$)

Assume:

- all $\mathcal{M}$-minimizations in $\mathcal{K}$ and all $\mathcal{E}$-minimizations in $\mathcal{K}^{\mathbf{op}}$ exist
- $\mathcal{K}$ has pullbacks along $\mathcal{M}$
- $\mathcal{K}$ has pushouts along $\mathcal{E}$

Proposition

For $C \in \mathcal{K}$, the following yield the same:

- $\mathcal{E}$-minimization (in $\mathcal{K}^{\mathbf{op}}$) of the $\mathcal{M}$-minimization of C (in $\mathcal{K}$)
- $\mathcal{M}$-minimization (in $\mathcal{K}$) of the $\mathcal{E}$-minimization of C (in $\mathcal{K}^{\mathbf{op}}$)

$$\downarrow$$

$\mathcal{K} = \mathrm{Coalg}_I(F)$ $\quad$ $F\colon \mathrm{Set} \circlearrowleft$

If F preserves inverse images, then reachability and observability minimization can be performed in any order.

Minimality Notions

$\mathcal{M}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$

$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$

Minimality Notions

$\mathcal{M}$-minimality of X	$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$
if $\mathcal{E} = \mathrm{Epi}$ & $\mathcal{K}$ has equalizers: $\Leftrightarrow$ all parallel $X \rightrightarrows Y$ equal	if $\mathcal{M} = \mathrm{Mono}$ & $\mathcal{K}$ has coequalizers: $\Leftrightarrow$ all parallel $Y \rightrightarrows X$ equal $\Leftrightarrow$ 'X subterminal'

Minimality Notions

$\mathcal{M}$-minimality of X	$\mathcal{E}$-minimality of X
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an iso	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an iso
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$
if $\mathcal{E} = \mathrm{Epi}$ & $\mathcal{K}$ has equalizers: $\Leftrightarrow$ all parallel $X \rightrightarrows Y$ equal	if $\mathcal{M} = \mathrm{Mono}$ & $\mathcal{K}$ has coequalizers: $\Leftrightarrow$ all parallel $Y \rightrightarrows X$ equal $\Leftrightarrow$ ‘X subterminal’

Minimization Criteria for..

- existence & uniqueness
- functoriality & adjointness
  $\mathcal{K}_{\mathrm{min}} \hookrightarrow \mathcal{K}$ coreflective

Interplay

For F-coalgebras:
if F preserves inverse images

References

[AMMS13] Jiří Adámek, Stefan Milius, Lawrence S. Moss, Lurdes Sousa. "Well-pointed Coalgebras". In: **Log. Methods Comput. Sci.** 9.3:2 (2013), 51 pp.

[Gum03] H. P. Gumm. **Algemeine Algebra. Mit einem Anhang über Universelle Coalgebra von H. P. Gumm**. Ed. by Thomas Ihringer. Vol. 10. Berliner Studienreihe zur Mathematik. Heldermann Verlag, 2003.

[Gum08] H. Peter Gumm. "On Minimal Coalgebras". In: **Applied Categorical Structures** 16.3 (June 2008), pp. 313–332. DOI: 10.1007/s10485-007-9116-1. URL: https://doi.org/10.1007/s10485-007-9116-1.

Simple Quotient in $\mathrm{Coalg}_I(F)$

Nothing new in $\mathrm{Coalg}_I(F)$ for $\twoheadrightarrow$

The forgetful functor

$$\mathrm{Coalg}_I(F) \longrightarrow \mathrm{Coalg}(F)$$

preserves and reflects simple coalgebras and simple quotients.

Proof

For every pointed coalgebra (C, c, i), we have isomorphic categories:

$$(C, c, i)/\mathrm{Coalg}_I(F) \quad \cong \quad (C, c)/\mathrm{Coalg}(F)$$

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)?

Assume: F preserves $\mathcal{M}$

Factorization of a homomorphism $h\colon (C, c) \to (D, d)$

$$\begin{tikzcd}
C \arrow[r, phantom] \arrow[rr, bend left, "h"] \arrow[d, "c"'] & & D \arrow[d, "d"] \\
FC \arrow[rr, bend right, "Fh"'] & & FD
\end{tikzcd}$$

[Standard]

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)?

Assume: F preserves $\mathcal{M}$

Factorization of a homomorphism $h\colon (C,c) \to (D,d)$

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] \arrow[rr, bend left, "h"] \arrow[d, "c"'] & \mathrm{Im}(h) \arrow[r, tail, "m"] & D \arrow[d, "d"] \\
FC \arrow[rr, bend right, "Fh"'] & & FD
\end{tikzcd}$$

[Standard]

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)?

Assume: F preserves $\mathcal{M}$

Factorization of a homomorphism $h\colon (C, c) \to (D, d)$

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] \arrow[rr, bend left, "h"] \arrow[d, "c"'] & \mathrm{Im}(h) \arrow[r, tail, "m"] & D \arrow[d, "d"] \\
FC \arrow[r, "Fe"] \arrow[rr, bend right, "Fh"'] & F\mathrm{Im}(h) \arrow[r, tail, "Fm"] & FD
\end{tikzcd}$$

[Standard]

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)?

Assume: F preserves $\mathcal{M}$

Factorization of a homomorphism $h\colon (C, c) \to (D, d)$

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] \arrow[d, "c"'] \arrow[rr, bend left, "h"] & \mathrm{Im}(h) \arrow[r, tail, "m"] \arrow[d, dashed, "\exists! u"] & D \arrow[d, "d"] \\
FC \arrow[r, "Fe"] \arrow[rr, bend right, "Fh"'] & F\mathrm{Im}(h) \arrow[r, tail, "Fm"] & FD
\end{tikzcd}$$

[Standard]

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)?

Assume: F preserves $\mathcal{M}$

Factorization of a homomorphism $h\colon (C,c) \to (D,d)$

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] \arrow[d, "c"'] \arrow[rr, bend left, "h"] & \mathrm{Im}(h) \arrow[r, tail, "m"] \arrow[d, dashed, "\exists! u"] & D \arrow[d, "d"] \\
FC \arrow[r, "Fe"] \arrow[rr, bend right, "Fh"'] & F\mathrm{Im}(h) \arrow[r, tail, "Fm"] & FD
\end{tikzcd}$$

[Standard]

$\Longrightarrow$ Not enough for a factorization system in Coalg(F)

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)!

Proposition: Diagonal fill-in in Coalg(F), if F preserves $\mathcal{M}$

$$\begin{tikzcd}
(A,a) \arrow[r, two heads, "e"] \arrow[d, "f"'] & (B,b) \arrow[d, "g"] \arrow[dl, dashed, "\exists! u"'] \\
(C,c) \arrow[r, tail, "m"] & (D,d)
\end{tikzcd}$$

[New]

$(\mathcal{E}$-carried, $\mathcal{M}$-carried$)$-factorization system in $\mathsf{Coalg}(F)$!

Proposition: Diagonal fill-in in $\mathsf{Coalg}(F)$, if F preserves $\mathcal{M}$

$$\begin{tikzcd}
(A,a) \arrow[r, two heads, "e"] \arrow[d, "f"'] & (B,b) \arrow[d, "g"] \arrow[dl, dashed, "\exists! u"'] \\
(C,c) \arrow[r, tail, "m"] & (D,d)
\end{tikzcd}$$

[New]

Proof. Obtain u as the diagonal in $\mathcal{C}$. It is a coalgebra morphism!

($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in Coalg(F)!

Proposition: Diagonal fill-in in Coalg(F), if F preserves $\mathcal{M}$

$$
\begin{tikzcd}
(A,a) \arrow[r, two heads, "e"] \arrow[d, "f"'] & (B,b) \arrow[d, "g"] \arrow[dl, dashed, "\exists! u"'] \\
(C,c) \arrow[r, tail, "m"] & (D,d)
\end{tikzcd}
$$

[New]

Proof. Obtain u as the diagonal in $\mathcal{C}$. It is a coalgebra morphism!

$$
\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[d, "a"'] \arrow[dr, "f"] & & B \arrow[dl, "u"'] \arrow[d, "g"] \arrow[dr, "b"] & \\
FA \arrow[r, phantom, "\text{f hom.}"] \arrow[d, "Ff"'] & C \arrow[r, tail, "m"] \arrow[dl, "c"] \arrow[dr, phantom, "\text{m hom.}"] & D \arrow[r, phantom, "\text{g hom.}"] \arrow[d, "d"] & FB \arrow[dl, "Fg"] \\
FC \arrow[rr, tail, "Fm"] & & FD &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[d, "a"'] \arrow[dr, phantom, "\text{e hom.}"] & & B \arrow[dl, "b"] \arrow[d, "b"] \\
FA \arrow[r, "Fe"] \arrow[d, "Ff"'] & FB \arrow[r, phantom, "\text{trivial}"] \arrow[dl, "Fu"] \arrow[dr, "Fg"'] & FB \arrow[d, "Fg"] \\
FC \arrow[rr, tail, "Fm"] & & Fm
\end{tikzcd}
$$

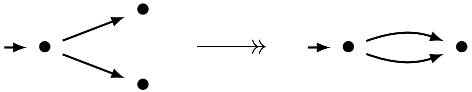
Uniqueness of the diagonal fill-in $\Rightarrow c \cdot u = Fu \cdot b$ $\square$

Tree unravelling as minimization

Setting

- $FX = \mathcal{B}X$ = bags = finite multisets
- Category $\mathcal{K}$: reachable pointed $\mathcal{B}$-coalgebras
- $(\mathsf{Mor}, \mathsf{Iso})$-factorization system on $\mathcal{K}$
- Mor-minimization = tree unravelling

Unravelling of siblings



Unravelling of a loop