# Supported Sets

## A New Foundation For Nominal Sets And Automata

Radboud University, Nijmegen

CMCS Short Contribution

ETAPS, Munich, April 02, 2022

## Nominal Sets

- ☺ Freshness & Binding of Names
- ☺ Initial Algebras: e.g. Lambda-Expressions mod. $\equiv_\alpha$
- ☺ (Final) Coalgebras:
  - Infinite Lambda-Trees (mod. $\equiv_\alpha$)
  - Automata with name binding
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## Drawbacks

- ↯ finitely presentable = orbit-finite $\neq$ finite
- ↯ not monadic over Set

Eilenberg-Moore category for a monad

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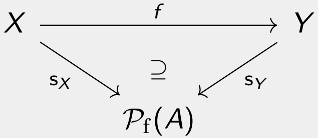

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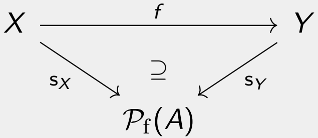


## Examples Objects

- the set $A$ itself, $\mathsf{s}_A = \mathrm{id}_A$
- singleton $\{a\}$ (for $a \in A$)
- every nominal set $X$ for $\mathsf{s}_X := \mathrm{supp}$

## Categorical Properties of Supp($A$)

- ☺ finitely presentable = finite (and Supp($A$) is lfp)
- ☺ (co)complete, cartesian closed
- ☺ $U\colon \mathrm{Supp}(A) \to \mathrm{Set} \quad \dashv \quad J\colon\ \mathrm{Set} \hookrightarrow \mathrm{Supp}(A)$
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## Name Binding via de Bruijn indices

$[\mathbb{A}]\colon \mathsf{Supp}(\mathbb{A}) \to \mathsf{Supp}(\mathbb{A})$ for $A := \mathbb{A} = \{a_0, a_1, \ldots\}$

$$[\mathbb{A}]X = X \qquad \mathsf{s}_{[\mathbb{A}]X}(x) := \{a_k \mid a_{k+1} \in \mathsf{s}_X(x), k \in \mathbb{N}\}$$

## Intuitive Definition $\mathsf{Nom}(M)$ for $M \subseteq (A \to A, \circ, \mathrm{id}_A)$

Nominal $M$-set = monoid action for $M$ + finite support

## Instances:

| *Order Symmetry* | *Equality Symmetry* | *Renaming Sets* |
| --- | --- | --- |
| $\mathsf{Nom}(\mathsf{Aut}(\mathbb{Q}, <))$ | $\mathsf{Nom}(\mathsf{Perm}_\mathsf{f}(\mathbb{A}))$ | $\mathsf{Nom}(\mathsf{Fin}(\mathbb{A}))$ |

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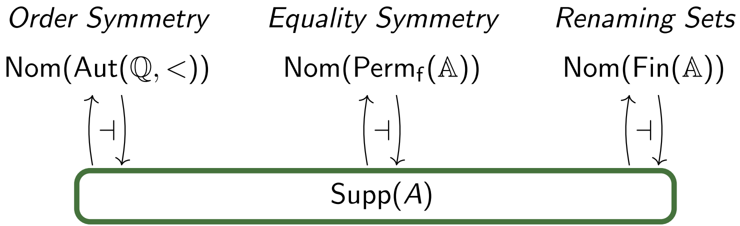
## Instances: Monadic adjunctions ($A$ countably infinite)



## Application: finite representation of nominal $M$-sets

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## Application: Generalized determinization

$\mathcal{EM}(T)$ →

$$
\begin{array}{ccc}
\mathrm{Nom}(M) & \xrightarrow{\bar{H}} & \mathrm{Nom}(M) \\
U \downarrow & & \downarrow U \\
\mathrm{Supp}(A) & \xrightarrow{H} & \mathrm{Supp}(A)
\end{array}
\quad \Longrightarrow \quad
\begin{array}{c}
\mathrm{Coalg}(\bar{H}) \\
\uparrow T \\
\mathrm{Coalg}(HT)
\end{array}
$$

Configuration Space

State Space

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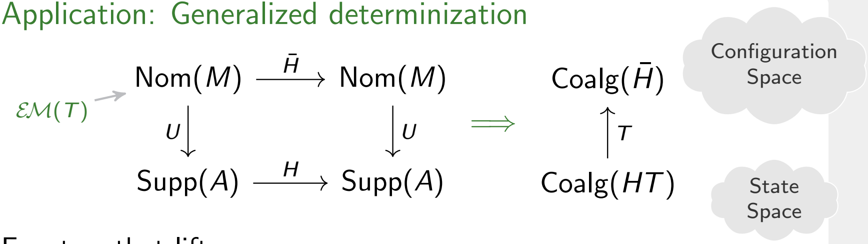


Functors that lift:

- $\times$, $+$, $\mathcal{P}_{\mathrm{f}}$, $\mathcal{P}_{\mathrm{ufs}}$, ...
- Binding functor $[\mathbb{A}]$ (for $A := \mathbb{A}$) ← Lifts to Gabbay & Pitts' abstraction functor
- E.g. RNNA $HX := 2 \times \mathcal{P}_{\mathrm{ufs}}([\mathbb{A}](-) + \mathbb{A} \times (-))$
- E.g. Register Automata $HX := 2 \times [\mathbb{A}]\mathcal{P}_{\mathrm{f}}(\Sigma\mathbb{A} \times X)$

## Relation to Presheaves

$$\mathsf{Nom}(\mathsf{Perm}_\mathsf{f}(\mathbb{A})) \underset{}{\overset{\top}{\rightleftarrows}} [|\mathcal{P}_\mathrm{f} A|, \mathsf{Set}]$$

Kurz, Petrisan, Velebil '10

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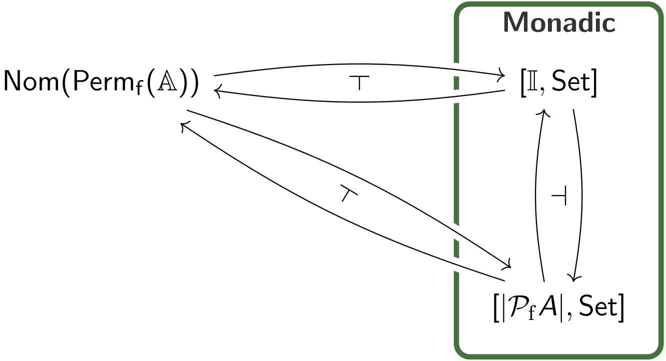
$\mathbb{I} = \mathcal{P}_\mathrm{f}(\mathbb{A})$ with all injective maps

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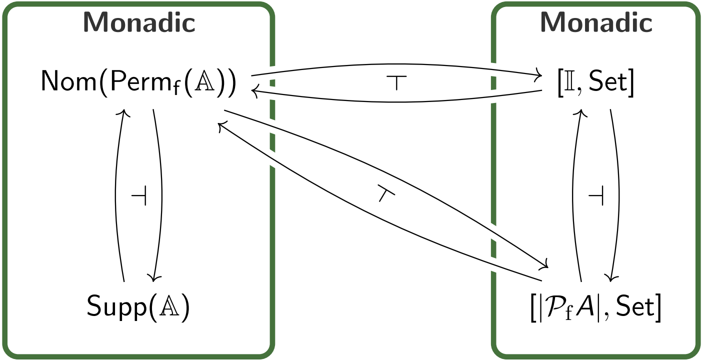


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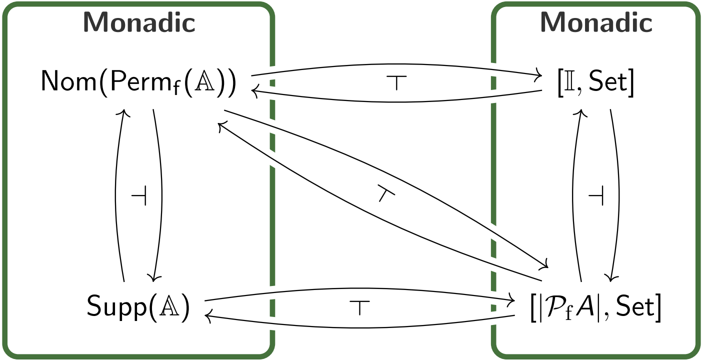


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References

[KPV10] Alexander Kurz, Daniela Petrisan, Jiri Velebil. "Algebraic Theories over Nominal Sets". In: **CoRR** abs/1006.3027 (2010). URL: http://arxiv.org/abs/1006.3027.